ANALYSIS OF TWO CLASSES OF CROSS DIFFUSION SYSTEMS

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Dedicated to

my parents

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Abstract

A mathematical and numerical analysis has been carried out for two cross diffusion systems arising in applied mathematics. The first system appears in modelling the movement of two interacting cell populations whose kinetics are of competition type. The second system models axial segregation of a mixture of two different granular materials in a long rotating drum. A fully practical piecewise linear finite element approximation for each system is proposed and studied. With the aid of a fixed point theorem, existence of the fully discrete solutions is shown. By using entropy-type inequalities and compactness arguments, the convergence of the approximation of each system is proved and hence existence of a global weak solution is obtained. Providing further regularity of the solution of the axial segregation model, some uniqueness results and error estimates are established. The long time behaviour of both systems is investigated and estimates between the weak solutions and the mean integrals of the corresponding initial data are derived. Finally, a practical algorithm for computing the numerical solutions of each system is described and some numerical experiments are performed to illustrate and verify the theoretical results.

Declaration

The work in this thesis is based on research carried out at the Numerical Analysis Group, the Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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Chapter 1

Introduction

In recent years much attention has been paid to the study of cross diffusion systems in the field of strongly coupled parabolic equations. Cross diffusion, which is only of relevance in multi-species models, was defined in Okubo, [54] page 170, as the diffusion of one type of species due to the presence of another species. Mathematically, cross diffusion occurs when the diffusion matrix of a system of partial differential equations is not strictly diagonal (see Murray [52] page 11). We see some examples later in this introduction. In mathematical biology applications, cross diffusion systems arise to model segregation phenomena between two competing species and are often expected to be relevant to the classical model of Lotka [49] and Volterra [65] for the interaction between a predator, u, and its prey, v,:

$$\frac{du}{dt} = \alpha uv - \beta u,$$
$$\frac{dv}{dt} = \delta v - \gamma uv,$$

where the non-negative parameters α, β, δ and γ represent the interaction of the two species.

To illustrate and understand the meaning of cross diffusion we give an example of a Lotka-Volterra type cross diffusion model. Let $S_1(x,t)$, $S_2(x,t)$ be the population densities of predator and prey, then the usual Lotka-Volterra predator-prey model with diffusion can be written as

$$\begin{aligned}
\frac{\partial S_1}{\partial t} &= D_1 \frac{\partial^2 S_1}{\partial x^2} - a_1 S_1 + b_1 S_1 S_2, \\
\frac{\partial S_2}{\partial t} &= D_2 \frac{\partial^2 S_2}{\partial x^2} + a_2 S_2 - b_2 S_1 S_2,
\end{aligned}$$

where D_1 and D_2 are the diffusivities of the two populations, a_1 and a_2 are the rates of death and birth of the individual species and finally the parameters b_1 and b_2 are the growth and decay factors due to binary interactions.

The above model can be modified to include cross diffusion terms that express the population flux of each species due to the presence of the another species (see Okubo [54]) :

$$\begin{aligned}
\frac{\partial S_1}{\partial t} &= D_1 \frac{\partial^2 S_1}{\partial x^2} + D_{12} \frac{\partial^2 S_2}{\partial x^2} - a_1 S_1 + b_1 S_1 S_2, \\
\frac{\partial S_2}{\partial t} &= D_2 \frac{\partial^2 S_2}{\partial x^2} + D_{21} \frac{\partial^2 S_1}{\partial x^2} + a_2 S_2 - b_2 S_1 S_2,
\end{aligned}$$

where the cross diffusion constants D_{12} and D_{21} can be positive, negative or zero. Positive cross diffusion means that one type of species tends to move in the direction of the lower density of the other type and vice versa. For instance, if a predator tends to diffuse in the direction of higher concentrations of prey and the prey tends to diffuse in the direction of lower concentrations of its predator, as expected, we assume $D_{12} < 0$ and $D_{21} > 0$. Of course, D_{12} or D_{21} may vanish for a non-responsive predator or non-motile prey. For further details about this model, see [54] Section (10.3.3). See [52] Section (1.2), for a description of another example of cross diffusion.

For the concepts of diffusion and cross diffusion, and their backgrounds and applications we refer to, e.g., [54], [52], [25], [50] and [64] and the references cited therein. For some earlier work on modelling cross diffusion systems see [60] and [54]. For more recent work on modelling cross diffusion systems see [3], [28], [55], [44] and [39]. We also refer to [33], [21], [9], [51] and [67] for some mathematical studies of a number of cross diffusion models of Lotka-Volterra type. Other mathematical studies of cross diffusion systems can be found in the literature, see for example [34], [20], [40], [45] and [15].

In this thesis, we use the finite element method as a technique to study two classes of strongly coupled cross diffusion systems arising in certain biological and physical applications. The first is a population model of competition type arising in biological study of the movement of two interacting cell populations. The second is an axial segregation model arising in physical study of granular materials.

1.1 Introduction to the population model

The scenarios of the movement of two interacting cell populations vary dependent on the details of the cells behaviour and other environmental factors. When individual cells in each population are widely separated, the movement of interacting cell populations is often represented simply via independent linear diffusion of each population (Painter and Sherratt [55] page 327). However, when cells are close enough for regular contacts, those of one type will influence the movement of the other cell population. Thus, as the cell density increases, cell-cell interactions will effect movement. One biological question is: how does the total cell density effect the movement properties of the cells? In fact, there are some mechanisms that may lead to dispersal of the population. One of these mechanisms is when cells detect and respond to a local gradient in the cell density. In such a movement, in one dimension space, the appropriate model for the dynamics of two cell populations is (see Painter and Sherratt [55]):

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\,\frac{\partial u}{\partial x} + \chi\, u\,\frac{\partial}{\partial x}(u+v) \right), \tag{1.1.1a}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(D\,\frac{\partial v}{\partial x} + \chi\, v\,\frac{\partial}{\partial x}(u+v) \right), \tag{1.1.1b}$$

where u and v are the densities of the two cell populations. χ is assumed to be a positive constant to ensure that cells move down gradients in the total density, i.e. to ensure that u and v move in the direction of lower concentrations of the total cell density $(u+v)$. This constant can be eliminated by rescaling the variables u and v, but for simplicity we choose $\chi = 1$. The diffusion coefficient D is non-negative where $D > 0$ implies that cells move down gradients of their own density, and $D = 0$ if the cells respond only to the total density gradient (see [55] for more details).

In the above model, the terms $\frac{\partial}{\partial x}\left(D\,\frac{\partial u}{\partial x}\right)$ and $\frac{\partial}{\partial x}\left(D\,\frac{\partial v}{\partial x}\right)$ are diffusion terms. The terms $\frac{\partial}{\partial x}\left(u\,\frac{\partial u}{\partial x}\right)$ and $\frac{\partial}{\partial x}\left(v\,\frac{\partial v}{\partial x}\right)$ are called self-diffusion terms. Finally, the terms $\frac{\partial}{\partial x}\left(u\,\frac{\partial v}{\partial x}\right)$ and $\frac{\partial}{\partial x}\left(v\,\frac{\partial u}{\partial x}\right)$ are cross diffusion terms. It is indicated in the literature that cross diffusion seems to create pattern formation whereas diffusion and self-diffusion tend to prevent pattern formation (see Lou and Ni [50]).

In the case of interacting two cell populations whose kinetics are of competition type, the model (1.1.1a)-(1.1.1b) will include predator-prey reaction terms representing the competitive situation. Assuming that the v cells have a competitive advantage over the u cells, the appropriate model can be written as (see Painter and Sherratt [55]):

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\,\frac{\partial u}{\partial x} + u\,\frac{\partial}{\partial x}(u+v)\right) + u\,(\,1 - u - v\,),$$
$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(D\,\frac{\partial v}{\partial x} + v\,\frac{\partial}{\partial x}(u+v)\right) + v\,(\,\gamma - u - v\,),$$

where the constant $\gamma > 1$ reflects the competitive advantage of the v cells. A specific instance to which this model could be applied is early tumour growth. For more details on the biological background, we refer the reader to [55] and the references therein.

In the first part of this thesis, we will consider mathematical aspects of the multi-dimensional version of the above model with homogeneous Neumann boundary conditions and appropriate initial data.

Let Ω be an open bounded domain in $\mathbb{R}^d\,(d \le 3)$ with a Lipschitz boundary $\partial\Omega$. We consider a nonlinear system of cross diffusion partial differential equations modelling the movement of two interacting cell populations whose kinetics are of competition type:

(P) Find $\{u(x,t), v(x,t)\} \in \mathbb{R}^{\ge 0} \times \mathbb{R}^{\ge 0}$ such that

$$\frac{\partial u}{\partial t} = \nabla \cdot (\,D\,\nabla u + u\,\nabla(u+v)) + f(u,v) \qquad \text{in } \Omega_T, \tag{1.1.2a}$$
$$\frac{\partial v}{\partial t} = \nabla \cdot (\,D\,\nabla v + v\,\nabla(u+v)) + g(u,v) \qquad \text{in } \Omega_T, \tag{1.1.2b}$$

with boundary conditions

$$\begin{aligned} &[\,D\,\nabla u + u\,\nabla(u+v)]\cdot\nu = 0 \\ &[\,D\,\nabla v + v\,\nabla(u+v)]\cdot\nu = 0 \end{aligned} \qquad \text{on } \partial\Omega\times(0,T), \tag{1.1.2c}$$

and initial conditions

$$u(x,0) = u^0(x)\,, \quad v(x,0) = v^0(x) \qquad \forall\, x\in\Omega, \tag{1.1.2d}$$

where $\Omega_T := \Omega \times (0,T)$, $T > 0$ and ν denotes the outward unit normal to $\partial\Omega$. As hinted above, the functions u and v are the densities of the two cell types. D is non-negative diffusion coefficient, and for the analysis of the problem we assume $D > 0$. The nonlinear predator-prey reaction terms are given as:

$$\begin{aligned} f(u,v) &= u\,(\,1-u-v\,),\\ g(u,v) &= v\,(\,\gamma-u-v\,), \end{aligned}$$

where the competition coefficient $\gamma > 1$ represents a growth advantage of v over u.

The parabolic system (1.1.2a)-(1.1.2b) is strongly coupled with full and non-symmetric diffusion matrix

$$A = \begin{pmatrix} D+u & u \\ v & D+v \end{pmatrix}.$$

We also not that there are values of $D > 0$ and $u, v \geq 0$ for which A is not positive definite. For this kind of strongly coupled system, it is well known that there is no general abstract theory that can be applied directly to obtain existence results (see [20]). Therefore, one must find an alternative bespoke approach to deal with the problem (P).

1.2 Introduction to the axial segregation model

In the second part of the thesis we consider the following cross diffusion parabolic system:

(Q) Find $\{w(x,t), z(x,t)\} \in [-1,1] \times \mathbb{R}$ such that

$$\frac{\partial w}{\partial t} = \nabla \cdot \big(\rho \nabla w - (1-w^2) \nabla z \big) \qquad \text{in } \Omega_T, \tag{1.2.1a}$$

$$\frac{\partial z}{\partial t} = \nabla \cdot (\nabla z + \lambda \nabla w) + \mu w - z \qquad \text{in } \Omega_T, \tag{1.2.1b}$$

where $\Omega = (0, L) \subset \mathbb{R}$ for some $L > 0$, and $\Omega_T := \Omega \times (0, T)$ for positive time $T > 0$. Together with Neumann-type boundary conditions

$$\begin{aligned} \big[\rho \nabla w - (1-w^2) \nabla z\big] (0, \cdot) &= [\nabla z + \lambda \nabla w] (0, \cdot) = 0, \\ \big[\rho \nabla w - (1-w^2) \nabla z\big] (L, \cdot) &= [\nabla z + \lambda \nabla w] (L, \cdot) = 0, \end{aligned} \qquad \text{in } (0, T), \tag{1.2.1c}$$

and initial conditions

$$w(x, 0) = w^0(x), \quad z(x, 0) = z^0(x) \qquad \forall\, x \in \Omega, \tag{1.2.1d}$$

where, for consistency of notation with the previous model, we write ∇ instead of $\frac{\partial}{\partial x}$.

The above system models axial separation of a mixture of two sorts of particles, A_1 and A_2, in a long rotating drum with length $L > 0$. Here $w = w_{A_1} - w_{A_2} \in [-1, 1]$ is the relative concentration of the mixture, where $w_{A_1}, w_{A_2} \in [0, 1]$ are the concentrations of the two particles A_1 and A_2. The variable z represents the so-called dynamic angle of repose which is defined as the angle of the slope of the free surface of grains in the drum as they flow continuously. The constant $\rho > 0$ is related to the Fick diffusion constants arising in the surface fluxes of the two materials, while the positive constant $\lambda > 0$ is proportional to the difference of the Fick diffusivities. Finally, the non-negative constant $\mu \geq 0$ is related to the static angle of repose of the particles.

Although cross diffusion equations are often considered in biological modelling, the cross diffusion model (Q) was proposed by Aranson *et al.* [3] in their study of the evolution of the relative concentration and the dynamic repose of a mixture of two different granular materials in a long rotating drum. From the physical point of view, mixtures of grains with different sizes in a long rotating drum exhibit both radial and axial size segregation. During the first few revolutions of the drum, radial segregation occurs and is often followed by slow axial segregation which leads to either a stable array of concentration bands or, after a very long time, to complete segregation (see [4]). For a fuller discussion on the background and derivation of the model we refer to [3], [4], [5], [34] and [43].

We note that the cross diffusion in the system (1.2.1a)-(1.2.1b) is represented by the terms $\nabla \cdot (\,(1-w^2)\,\nabla z\,)$ and $\lambda\,\nabla^2 w$. Furthermore, as for the problem (P), the system (1.2.1a)-(1.2.1b) is strongly coupled with full and non-symmetric diffusion matrix

$$B = \begin{pmatrix} \rho & -(1-w^2) \\ \lambda & 1 \end{pmatrix},$$

which is generally not positive definite.

1.3 Research objectives and methodology

The work in this thesis will consist of three main parts:

(1) Analysis of the problem (P).

(2) Analysis of the problem (Q).

(3) Numerical experiments for (P) and (Q).

In the first part, Chapters 2, 3 and 4, we provide an extended study of the problem (P). As a main objective, we study the existence of a global weak solution of the system (1.1.2a)-(1.1.2d). An efficient method to do that is by introducing and analyzing a fully discrete finite element approximation of (P). The main features of the system will be reflected explicitly in the analysis of the fully approximation problem. For this reason, the need to derive an entropy inequality "energy estimate" of the problem (P) is the key in the analysis of the approximation problem. The entropy inequality of the problem (P) can be made by testing the equations (1.1.2a) and (1.1.2b) with $\ln u$ and $\ln v$ respectively. However, this will require us to go through a regularization procedure in order that we treat the singular nature of the derived inequality in the region $\mathbb{R}^{\leq 0}$. Hence, a well defined entropy inequality of the regularization problem of (P) can be established and uniform bounds on the regularized functions, independent of the regularization parameter, can be obtained. The entropy inequality and the uniform bounds of the regularized problem provide the foundation of a discrete analogue entropy inequality and uniform estimates of the corresponding approximation problem. Such estimates are needed to prove the convergence of the regularized fully approximated problem as the regularization parameter and the discretization parameters simultaneously tend to zero, and therefore we obtain existence of a weak solution to the system (1.1.2a)-(1.1.2d).

To sum up, the finite element approach used to show the existence of a non-negative global weak solution of (P) consists of four main steps. Firstly, we introduce a regularized problem of (P) and establish its entropy inequality. Secondly, we consider a fully discrete finite element approximation of the regularized problem and prove the existence of the approximate solutions at each time step using appropriate initial data. Thirdly, we derive a discrete analogue entropy inequality and obtain some bounds of the approximate solutions. Finally, we study the convergence of the fully approximation problem.

Unfortunately, the lack of H^1-norm bounds in the problem (P) will prevent the proof of convergence in the final step. Indeed, this will be treated successfully using a crucial idea, where we consider an alternative "equivalent" problem to (P), that gives us the necessary bounds to prove the convergence.

The second part of the thesis, Chapter 6, will be devoted to the analysis of the system (1.2.1a)-(1.2.1d). As both systems (1.1.2a)-(1.1.2d) and (1.2.1a)-(1.2.1d) belong to a similar class of equations, the analysis of problem (P) will significantly contribute to our study of the problem (Q). In particular, similar arguments used for (P) will be employed to prove the existence of a global weak solution of the system (1.2.1a)-(1.2.1d). Due to the structure of (Q), the second part of this thesis will also involve a discussion of the uniqueness of the weak solution of (Q) as well as a derivation of some fully discrete error estimates.

In the third part of the thesis, Chapters 5 and 7, we perform some programming in Fortran and Matlab to verify the established theoretical results in the first two parts.

The idea of defining and exploiting an entropy inequality has been used in the study of different types of partial differential equations, see [10] and [12], where a thin film equation is studied, and [33], [34], [21], [9] and [22] where cross diffusion systems are considered. The approach adopted in this thesis uses the standard piecewise linear finite element method. For references that use this approach, or employ similar arguments and tools to our own, see for example [7], [9], [10], [11], [12], [35] and [63]. For the theoretical tools, techniques and results used in this thesis see e.g. [1], [46], [23], [57], [58] and [32]. Below we give a brief description of the content of each chapter of the thesis.

In Chapter 2, the population model (1.1.2a)-(1.1.2d) is considered. A truncated alternative "equivalent" solvable problem to (P) is introduced. A regularized problem of the truncated system is studied and some *a priori* estimates of the regularized functions are obtained. A practical fully discrete approximation of the regularized problem is presented using a finite element method, with piecewise linear basis functions, to discretise in space and using backward Euler method to discretise in time. Then, some technical lemmata necessary for the analysis of the approximate problem are discussed. Finally, existence of the approximate solution at each time level is proven using the Schauder's fixed point theorem.

In Chapter 3, the analysis of the population model (1.1.2a)-(1.1.2d) is continued. Some stability bounds on the fully discrete approximations, defined in Chapter 2, are derived. Using classical compactness arguments, the convergence of the approximate problem to (P) is studied. Existence of a global weak solution of the system (1.1.2a)-(1.1.2d) is shown.

In Chapter 4, improved results for the system (1.1.2a)-(1.1.2d) are achieved by considering a "fully" truncated alternative problem to (P). In the absence of the reaction terms, further features of the system (1.1.2a)-(1.1.2d) are explored.

In Chapter 5, Some practical algorithms for computing the numerical solutions of problem (P) are described. Some numerical simulations in one space dimension are performed and discussed.

In Chapter 6 , the axial segregation model (1.2.1a)-(1.2.1d) is considered. A regularized fully discrete finite element approximation of the problem (Q) is studied. Existence and uniqueness of the approximations are established. By studying the convergence of the fully discrete approximate problem, existence of a global weak solution of the system (1.2.1a)-(1.2.1d) is shown. The uniqueness of the derived weak solution is discussed. Furthermore, an error bound between the fully discrete and weak solutions is studied. Finally, the long time behaviour of the solutions of the system (1.2.1a)-(1.2.1d) is discussed and an estimate between each variable and its mean integral is derived.

In Chapter 7, a practical algorithm for solving the finite element problem of (Q) at each time step is introduced. Some numerical results are presented to illustrate the segregation behaviour.

Finally, in Chapter 8, some conclusion remarks are given and some possible future work are addressed.

Chapter 2

The population model: A fully discrete approximation of a regularized truncated problem

In Section 2.1 we briefly review some basic notation, definitions and tools that will be used throughout the thesis. In Section 2.2 we introduce a truncated alternative problem to (P). In Section 2.3 we introduce a regularized problem of the truncated system. Then we obtain some *a priori* estimates of the regularized functions, independent of the regularization parameter, via deriving a well defined entropy inequality of the regularized problem. In Section 2.4 we present some finite element notation which will be used in the current and the following chapters. We propose a practical fully discrete finite element approximation of the regularized problem and present some necessary lemmata. Finally, we use a fixed point theorem to show the existence of the approximate solutions.

2.1 Notation

Let G be a bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$, with boundary ∂G. For $d = 2, 3$ we assume that ∂G is a Lipschitz boundary. Throughout this thesis we use the usual

Sobolev spaces $W^{m,p}(G)$, $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$, which are defined by

$$W^{m,p}(G) := \{u \in L^p(G) : D^\alpha u \in L^p(G) \ \text{ for } 0 \le |\alpha| \le m\},$$

with the associated norms and semi-norms given, respectively, by

$$\|u\|_{m,p,G} := \begin{cases} \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha u\|_{0,p,G}^p \right)^{\frac{1}{p}} & \text{if } \ 1 \le p < \infty, \\ \max_{0 \le |\alpha| \le m} \|D^\alpha u\|_{0,\infty,G} & \text{if } \ p = \infty; \end{cases}$$

and

$$|u|_{m,p,G} := \begin{cases} \left(\sum_{|\alpha| = m} \|D^\alpha u\|_{0,p,G}^p \right)^{\frac{1}{p}} & \text{if } \ 1 \le p < \infty, \\ \max_{|\alpha| = m} \|D^\alpha u\|_{0,\infty,G} & \text{if } \ p = \infty; \end{cases}$$

where D^α is the standard multi-index notation for the partial derivative of order $|\alpha|$ and

$$\|\eta\|_{0,p,G} := \begin{cases} \left(\int_G |\eta|^p dx \right)^{\frac{1}{p}} & \text{if } \ 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x \in G} |\eta(x)| & \text{if } \ p = \infty, \end{cases}$$

(e.g., see Adams [1] or Robinson [58]). For $m = 0$, the space $W^{0,p}(G)$ will be denoted by $L^p(G)$. In the case $p = 2$, the Hilbert space $W^{m,2}(G)$ will be denoted by $H^m(G)$ with the associated norm and semi-norm written as $\|\cdot\|_{m,G}$ and $|\cdot|_{m,G}$, respectively. For ease of notation, when $G \equiv \Omega$ the subscript "Ω" will be dropped on the above norms and semi-norms.

In our work, the usual $L^2(\Omega)$ inner product over Ω with the norm $\|\cdot\|_{0,2} \equiv \|\cdot\|_0$ is denoted by $(\cdot, \cdot)$. The dual space of a Banach space X is denoted by X', and we write $\langle \cdot, \cdot \rangle_{X',X}$ for the duality pairing between X' and X.

We also use function spaces depending on space and time. Let $1 \le p \le \infty$ and X be a Banach space. We denote $L^p(0, T; X)$ to be the Banach space that consists of all those functions $u(t) : (0, T) \to X$ *a.e.* such that $t \to \|u(t)\|_X$ in $L^p(0, T)$, with norm

$$\|u\|_{L^p(0,T;X)} := \begin{cases} \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}} & \text{if } \ 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{t \in (0,T)} \|u(t)\|_X & \text{if } \ p = \infty. \end{cases}$$

For ease of notation, we write the commonly used time-dependent space $L^p(0,T;L^p(\Omega))$ as $L^p(\Omega_T)$.

Furthermore, we define $C([0,T];X)$, the space of continuous functions from $[0,T]$ into X, which consists of those $u(t) : [0,T] \to X$ such that $u(t) \to u(t_0)$ in X as $t \to t_0$. We recall that $C([0,T];X)$ is a Banach space with the associated norm (see [62] page 43):

$$\|u\|_{C([0,T];X)} := \sup_{t \in [0,T]} \|u(t)\|_X \, .$$

For later purposes, we recall the Sobolev interpolation theorem (see Adams [2]): Let $u \in W^{m,p}(\Omega)$, for $1 \leq p \leq \infty$ and $m \geq 1$, then there are constants C and $\sigma = \frac{d}{m}\left(\frac{1}{p} - \frac{1}{r}\right)$ such that the following inequality holds

$$\|u\|_{0,r} \leq C \|u\|_{0,p}^{1-\sigma} \|u\|_{m,p}^{\sigma}, \qquad \text{for } r \in \begin{cases} [p,\infty] & \text{if } m - \frac{d}{p} > 0, \\ [p,\infty) & \text{if } m - \frac{d}{p} = 0, \\ [p, -\frac{d}{m-d/p}] & \text{if } m - \frac{d}{p} < 0. \end{cases} \tag{2.1.1}$$

We also need the following version of the Sobolev interpolation results (e.g. see [19]): Let $u \in H^1(\Omega)$ then there are constants C and $\theta = \frac{2\,d\,(r-1)}{r\,(d+2)}$ such that the following inequality holds

$$\|u\|_{0,r} \leq C \|u\|_{0,1}^{1-\theta} \|u\|_1^{\theta}, \qquad \text{for } r \in \begin{cases} [1,\infty] & \text{if } d = 1, \\ [1,\infty) & \text{if } d = 2, \\ [1,6] & \text{if } d = 3. \end{cases} \tag{2.1.2}$$

It will be useful in the work that follows to note the following well-known Sobolev embedding results (which can be seen immediately from the above interpolation inequalities) :

$$H^1(\Omega) \hookrightarrow L^r(\Omega), \qquad \text{holds for } r \in \begin{cases} [1,\infty] & \text{if } d = 1, \\ [1,\infty) & \text{if } d = 2, \\ [1,6] & \text{if } d = 3; \end{cases} \tag{2.1.3}$$

where "$\hookrightarrow$" denotes the continuous embedding. Further, we have from the Rellich-Kondrachov theorem, e.g. see [23] page 114 and [19] page 8, that the embedding in (2.1.3) is compact with $r \in [1,6]$ replaced by $r \in [1,6)$ in the case $d = 3$ [1]. The compact embedding will be denoted by the symbol "$\overset{c}{\hookrightarrow}$".

For later use, we recall the following embedding compactness result (see [48], page 58): Let X, Y and Z be three Banach spaces with X and Z being reflexive and $X \overset{c}{\hookrightarrow} Y \hookrightarrow Z$. Also let

$$W = \left\{ u : \; u \in L^r(0,T;X), \; \frac{\partial u}{\partial t} \in L^s(0,T;Z) \right\},$$

where $T < \infty$ and $1 < r, \; s < \infty$. Then

$$W \overset{c}{\hookrightarrow} L^r(0,T;Y). \tag{2.1.4}$$

We also require the Grönwall lemma both in its integral and differential form. For completeness we state the lemma and we refer to [31] for the proof of more general results. We start with the integral form:

Let β be a non-negative constant and let $u(t) \in L^\infty(0,T)$ and $v(t) \in L^1(0,T)$ be non-negative functions such that for *a.e.* $t \in (0,T)$

$$u(t) \le \beta + \int_0^t u(s)\, v(s) \; ds.$$

Then for *a.e.* $t \in (0,T)$

$$u(t) \le \beta \, \exp\left(\int_0^t v(s) \; ds \right). \tag{2.1.5}$$

[1] To deduce the compact embedding results for $d = 1$, we note the fact $H^1(\Omega) \overset{c}{\hookrightarrow} L^\infty(\Omega) \hookrightarrow L^r(\Omega)$, $r \in [1,\infty]$, (see [19], page 9).

We now state the differential form:

Let $f(t) \in W^{1,1}(0,T)$ and $g(t),\ h(t),\ w(t) \in L^1(0,T)$ be non-negative functions such that for *a.e.* $t \in (0,T)$

$$f'(t) + g(t) \le h(t) + f(t)\, w(t).$$

Then for *a.e.* $t \in (0,T)$

$$f(t) + \int_0^t g(s)\ ds \le e^{\Lambda(t)} f(0) + e^{\Lambda(t)} \int_0^t h(s)\ ds, \tag{2.1.6}$$

where $\Lambda(t) := \int_0^t w(s)\ ds$.

For later purposes, we recall the generalized version of the Hölder's inequality: Let $u_1 \in L^{p_1}(\Omega),\ u_2 \in L^{p_2}(\Omega)$ and $u_3 \in L^{p_3}(\Omega)$ such that $1 \le p_1,\ p_2,\ p_3 \le \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, then $u_1\, u_2\, u_3 \in L^1(\Omega)$ and

$$\int_\Omega |u_1\, u_2\, u_3| dx \le \left(\int_\Omega |u_1|^{p_1} dx \right)^{\frac{1}{p_1}} \left(\int_\Omega |u_2|^{p_2} dx \right)^{\frac{1}{p_2}} \left(\int_\Omega |u_3|^{p_3} dx \right)^{\frac{1}{p_3}}. \tag{2.1.7}$$

Another well-known inequality we need is the Poincaré inequality (e.g. see Wloka [66], page 117)

$$\|\eta\|_0^2 \le C_p \left(|\eta|_1^2 + |(\eta, 1)|^2 \right) \quad \forall \eta \in H^1(\Omega), \tag{2.1.8}$$

where C_p is a positive constant that depends on the domain Ω.

For completeness we also mention some elementary results which will be used later on. We make frequent use of the Young's inequality

$$a\, b \le \varepsilon^{p_1} \frac{a^{p_1}}{p_1} + \varepsilon^{-p_2} \frac{b^{p_2}}{p_2}, \qquad \frac{1}{p_1} + \frac{1}{p_2} = 1,$$

valid for any $a,\ b \ge 0,\ \varepsilon > 0$ and $p_1,\ p_2 > 1$.

We shall also need the following simple inequality

$$(a-b)^2 \ge \frac{a^2}{2} - b^2 \qquad \forall a,\, b \in \mathbb{R}, \tag{2.1.9}$$

which follows from a direct application of the Young's inequality.

Another useful consequence of the Young's inequality is the following

$$a\, b \ge -\frac{\varepsilon\, a^2}{2} - \frac{b^2}{2\, \varepsilon} \qquad \forall a,\, b \in \mathbb{R}\ \ \forall \varepsilon > 0. \tag{2.1.10}$$

Finally, we note the following elementary inequalities, valid for any $a \in \mathbb{R}$:

$$(1-a) = [1-a]_+ + [1-a]_- \leq [1-a]_+ \leq 1 - [a]_- \,, \tag{2.1.11}$$

$$(a-1) = [a-1]_+ + [a-1]_- \geq [a-1]_- \geq [a]_- - 1, \tag{2.1.12}$$

where $[a]_+ = \max\{a, 0\}$ and $[a]_- = \min\{a, 0\}$.

Throughout C represents a generic positive constant, independent of any regularization and discretization parameter, which may change from one expression to another. In addition, $C(c_1,\, c_2,\, \cdots, c_n)$ denotes a constant depending on $\{c_i\}_{i=1}^n$.

2.2 A truncated alternative problem

In this short, but important, section we make a significant step towards showing the existence of a global in-time weak solution of the problem (P). Our approach to prove existence is based on the idea of defining an entropy inequality that leads us to obtain energy estimates. One of the main difficulties of (P) is how to deal with the diffusion terms to derive H^1-norm bounds of the solutions u and v. To overcome this difficulty, we need to note that from a biological point of view one does not expect both densities, u and v, to be unbounded. Noting this and the advantage of the v cells over the u cells, it is convenient for the mathematical analysis of (P) to replace the term $u\, \nabla(u+v)$ in (1.1.2a) by $\phi(u)\, \nabla(u+v)$ and to replace the reaction terms $f(u, v)$ and $g(u, v)$ by $f_M(u, v)$ and $g_M(u, v)$, respectively, where

$$\phi(u) := [\, u - M\,]_- + M, \tag{2.2.1}$$

$$f_M(u, v) := u - \phi(u)\,(\, u + v\,), \tag{2.2.2}$$

$$g_M(u, v) := \gamma\, v - v\,(\, \phi(u) + v\,). \tag{2.2.3}$$

Here M is fixed positive number, and for later computational purposes we choose $M \geq e$. Without loss of generality, such a replacement can be considered even if v does not have advantage over u.

Thus the modified problem is:

($\mathbf{P}_M$) For fixed $M \geq e$, find $\{u(x,t), v(x,t)\} \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ such that

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(D \, \nabla u + \phi(u) \, \nabla(u+v)\right) + f_M(u,v) \qquad \text{in } \Omega_T, \tag{2.2.4a}$$

$$\frac{\partial v}{\partial t} = \nabla \cdot \left(D \, \nabla v + v \, \nabla(u+v)\right) + g_M(u,v) \qquad \text{in } \Omega_T, \tag{2.2.4b}$$

with homogeneous Neumann boundary conditions

$$\left.\begin{aligned} &[\, D \, \nabla u + \phi(u) \, \nabla(u+v)] \cdot \nu = 0 \\ &[\, D \, \nabla v + v \, \nabla(u+v)] \cdot \nu = 0 \end{aligned}\right. \qquad \text{on } \partial\Omega \times (0,T), \tag{2.2.4c}$$

and initial conditions

$$u(x,0) = u^0(x)\,, \quad v(x,0) = v^0(x) \qquad \forall\, x \in \Omega, \tag{2.2.4d}$$

where $\phi(u)$, $f_M(u,v)$ and $g_M(u,v)$ are defined by (2.2.1)-(2.2.3) above.

We mention that the functions u, v and $\phi(u)$ in the problem (P_M) should be written with a subscript "M", i.e. u_M, v_M and $\phi_M(u_M)$ respectively, but the subscript is dropped for ease of notation.

It is clear that the problem (P_M) is equivalent to the problem (P) if the number M is chosen large enough such that $u \leq M$ where u is solution to (P_M). This has meaning since we expect at least one of the densities to be bounded. Actually, it is well known in the biological literature that all densities are assumed to be bounded. This is not wasted, as improved results can be derived by considering an alternative problem to (P_M) (see the discussion in Chapter 4). The replacement employed above will play a crucial role in the study of problem (P) as it is the key to obtaining the needed bounds, on u and v, in the analysis. We indicate that the idea of considering an alternative solvable problem to (P) is inspired from an argument employed in [10] on the study of a thin film equation.

2.3 A regularized problem

The key step of our analysis in proving existence of a global weak solution of the system (2.2.4a)-(2.2.4d) is to derive *a priori* estimates. To achieve this, we use a mathematical approach that deals with an entropy inequality of a regularized problem of (P_M). Such an approach has been employed in studying different kinds of partial differential equations, e.g. see [9], [10], [11] and [12]. By using an appropriate entropy functional, we first obtain some *a priori* estimates on any positive solution of the model (P_M).

We define a function $F \in C^2(\mathbb{R}^{>0})$ such that $\phi(u)\, F''(u) = 1$ and $F(1) = 0$; that is $F : \mathbb{R}^{>0} \to \mathbb{R}^{\geq 0}$ given by

$$F(s) := \begin{cases} (\ln s - 1)\, s + 1 & \text{if } 0 < s \leq M\,, \\ \frac{s^2 - M^2}{2\,M} + (\ln M - 1)\, s + 1 & \text{if } s \geq M\,; \end{cases} \tag{2.3.1}$$

and hence,

$$F'(s) := \begin{cases} \ln s & \text{if } 0 < s \leq M\,, \\ \frac{s}{M} + \ln M - 1 & \text{if } s \geq M\,, \end{cases}$$

$$\text{and} \quad F''(s) := \begin{cases} \frac{1}{s} & \text{if } 0 < s \leq M\,, \\ \frac{1}{M} & \text{if } s \geq M\,. \end{cases}$$

We also define the function $G \in C^\infty(\mathbb{R}^{>0})$ satisfying $v\, G''(v) = 1\,;$ that is $G : \mathbb{R}^{>0} \to \mathbb{R}^{\geq 0}$ given by

$$G(s) := (\ln s - 1)\, s + 1; \tag{2.3.2}$$

and hence,

$$G'(s) = \ln s \qquad \text{and} \qquad G''(s) = \tfrac{1}{s}.$$

Assuming positive values of the population densities, u and v, one can define the non-negative entropy functional

$$E(t) = \int_\Omega \Big(F(u) + G(v) \Big)\, \mathrm{d}x\,,$$

with the corresponding entropy inequality

$$E(t)+\int_0^t \left(\tfrac{D}{M}\,||\nabla u||_0^2 + ||\nabla u + \nabla v||_0^2\right)\,\mathrm{d}t$$
$$\leq E(0) + \int_0^t \int_\Omega \left(f_M(u,v)\,F'(u) + g_M(u,v)\,G'(v)\right)\,\mathrm{d}x\ \mathrm{d}t, \qquad (2.3.3)$$

for $0 < t < T$. This can be derived by multiplying (2.2.4a) and (2.2.4b) by $F'(u)$ and $G'(v)$ respectively, integrating by parts over Ω and summing the resulting equations, after recalling (2.3.1), (2.3.2) and (2.2.4c). However, as the functions $F(u)$ and $G(v)$ are defined on $\mathbb{R}^{>0}$, the inequality (2.3.3) can be made rigorous only if both $u(x,t)$ and $v(x,t)$ are positive. To deal with the singularity on the non-positive part, we need to go through an appropriate regularization procedure.

For computational purposes, we replace the function $F \in C^2(\mathbb{R}^{>0})$ for any $\varepsilon \in (0, e^{-1})$ by the regularized function $F_\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ given by

$$F_\varepsilon(s) := \begin{cases} \frac{s^2-\varepsilon^2}{2\,\varepsilon} + (\ln\varepsilon - 1)\,s + 1 & \text{if } s \leq \varepsilon\,, \\ (\ln s - 1)\,s + 1 & \text{if } \varepsilon \leq s \leq M\,, \\ \frac{s^2-M^2}{2\,M} + (\ln M - 1)\,s + 1 & \text{if } s \geq M\,. \end{cases} \qquad (2.3.4a)$$

Therefore,

$$F'_\varepsilon(s) := \begin{cases} \frac{s}{\varepsilon} + \ln\varepsilon - 1 & \text{if } s \leq \varepsilon\,, \\ \ln s & \text{if } \varepsilon \leq s \leq M\,, \\ \frac{s}{M} + \ln M - 1 & \text{if } s \geq M\,; \end{cases} \qquad (2.3.4b)$$

$$F''_\varepsilon(s) := \begin{cases} \frac{1}{\varepsilon} & \text{if } s \leq \varepsilon\,, \\ \frac{1}{s} & \text{if } \varepsilon \leq s \leq M\,, \\ \frac{1}{M} & \text{if } s \geq M\,. \end{cases} \qquad (2.3.4c)$$

We also replace the function $\phi(s)$ by the regularized function $\phi_\varepsilon : \mathbb{R} \to [\varepsilon, M]$

defined by

$$\phi_\varepsilon(s) := [F_\varepsilon''(s)]^{-1} := \begin{cases} \varepsilon & \text{if } s \le \varepsilon\,, \\ s & \text{if } \varepsilon \le s \le M\,, \\ M & \text{if } s \ge M\,. \end{cases} \tag{2.3.5}$$

For later purposes, we recall the following properties concerning the functions $F_\varepsilon(s)$ and $\phi_\varepsilon(s)$:

- For all $\varepsilon \in (0, e^{-1})$ and for all $s \le 0$ we note that

$$F_\varepsilon(s) := \frac{s^2 - \varepsilon^2}{2\,\varepsilon} + (\ln \varepsilon - 1)\, s + 1 \ge \tfrac{s^2}{2\,\varepsilon}. \tag{2.3.6}$$

- For all $\varepsilon \in (0, e^{-1})$ and for all $s \ge 0$ we have that

$$F_\varepsilon(s) \ge \tfrac{s^2}{4\,M} - \tfrac{3\,M}{2}. \tag{2.3.7}$$

To show this, we note firstly for any $s \in [0,1]$ that $F_\varepsilon'(s) \le 0$ and hence

$$F_\varepsilon(s) \ge F_\varepsilon(1) = 0 \ge \tfrac{s^2}{4\,M} - \tfrac{3\,M}{2}.$$

Secondly, for $s \in [1, M]$ we have that

$$r(s) := \tfrac{1}{s} - \tfrac{1}{M} \ge 0 \implies \int_1^s \int_1^t r(u)\;\mathrm{d}u\,\mathrm{d}t \ge 0,$$

that is

$$(\ln s - 1)\, s + 1 - \tfrac{s^2}{2\,M} + \tfrac{s}{M} - \tfrac{1}{2\,M} \ge 0,$$

and hence for all $s \in [1, M]$

$$F_\varepsilon(s) \ge \tfrac{s^2}{2\,M} - \tfrac{s}{M} + \tfrac{1}{2\,M} \ge \tfrac{s^2}{4\,M} - \tfrac{3\,M}{2}.$$

Finally, for $s \ge M$ we have from the Young's inequality that

$$F_\varepsilon(s) \ge \tfrac{s^2}{2\,M} - \tfrac{M}{2} - s \ge \tfrac{s^2}{4\,M} - \tfrac{3\,M}{2}.$$

- Furthermore, it is a simple matter to show that for all $\varepsilon \in (0, e^{-1})$ and for all $s \in \mathbb{R}$

$$s\, F_\varepsilon'(s) \leq 2\, F_\varepsilon(s) + 1, \tag{2.3.8}$$

$$s\, F_\varepsilon'(s) \geq \phi_\varepsilon(s)\, F_\varepsilon'(s) \geq s - 1. \tag{2.3.9}$$

To see (2.3.8), define

$$J_\varepsilon(s) = 2\, F_\varepsilon(s) - s\, F_\varepsilon'(s) + 1 \;\Rightarrow\; J_\varepsilon'(s) = F_\varepsilon'(s) - s\, F_\varepsilon''(s)\,.$$

For $s \leq \varepsilon$, $J_\varepsilon'(s) = \ln \varepsilon - 1 \leq 0$ and so

$$J_\varepsilon(s) \geq J_\varepsilon(\varepsilon) = \varepsilon\,(\ln \varepsilon - 2) + 3 \geq 0\,.$$

For $\varepsilon \leq s \leq M$, $J_\varepsilon'(s) = \ln s - 1$. Hence, $J_\varepsilon'(s) = 0$ at $s = e \in [\varepsilon, M]$ and

$$J_\varepsilon(s) \geq J_\varepsilon(e) = 3 - e > 0\,.$$

For $s \geq M$, we have as $M \geq e$ that $J_\varepsilon'(s) = \ln M - 1 \geq 0$ and

$$J_\varepsilon(s) \geq J_\varepsilon(M) = M\,(\ln M - 2) + 3 \geq 0\,.$$

Thus we conclude (2.3.8) for all $s \in \mathbb{R}$.

The first inequality in (2.3.9) follows directly on noting that

$$\begin{array}{llll} \phi_\varepsilon(s) \geq s & \text{and} & F_\varepsilon'(s) \leq 0 & \forall s \leq \varepsilon\,, \\ \phi_\varepsilon(s) \leq s & \text{and} & F_\varepsilon'(s) \geq 0 & \forall s \geq M\,. \end{array}$$

On setting

$$Q_\varepsilon(s) := \phi_\varepsilon(s)\, F_\varepsilon'(s) - s + 1$$
$$= \begin{cases} F_\varepsilon(\varepsilon) & \text{if } s \leq \varepsilon\,, \\ F_\varepsilon(s) & \text{if } \varepsilon \leq s \leq M\,, \\ F_\varepsilon(M) & \text{if } s \geq M\,, \end{cases}$$

we have from the non-negativity of $F_\varepsilon(s)$ that $Q_\varepsilon(s) \geq 0$. So, the second inequality in (2.3.9) holds.

We also replace the function $G \in C^\infty(\mathbb{R}^{>0})$ for any $\varepsilon \in (0, e^{-1})$ by the regularized function $G_\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ where

$$G_\varepsilon(s) := \begin{cases} \frac{s^2-\varepsilon^2}{2\,\varepsilon} + (\ln \varepsilon - 1)\, s + 1 & \text{if } s \leq \varepsilon\,, \\ (\ln s - 1)\, s + 1 & \text{if } \varepsilon \leq s \leq \varepsilon^{-1}\,, \\ \frac{\varepsilon\,(s^2-\varepsilon^{-2})}{2} + (\ln \varepsilon^{-1} - 1)\, s + 1 & \text{if } s \geq \varepsilon^{-1}\,; \end{cases} \tag{2.3.10a}$$

and therefore,

$$G'_\varepsilon(s) := \begin{cases} \frac{s}{\varepsilon} + \ln \varepsilon - 1 & \text{if } s \leq \varepsilon\,, \\ \ln s & \text{if } \varepsilon \leq s \leq \varepsilon^{-1}\,, \\ \varepsilon\, s + \ln \varepsilon^{-1} - 1 & \text{if } s \geq \varepsilon^{-1}\,, \end{cases} \tag{2.3.10b}$$

$$G''_\varepsilon(s) := \begin{cases} \frac{1}{\varepsilon} & \text{if } s \leq \varepsilon\,, \\ \frac{1}{s} & \text{if } \varepsilon \leq s \leq \varepsilon^{-1}\,, \\ \varepsilon & \text{if } s \geq \varepsilon^{-1}\,. \end{cases} \tag{2.3.10c}$$

For all $\varepsilon \in (0, e^{-1})$ we define the function $\psi_\varepsilon : \mathbb{R} \to [\varepsilon, \varepsilon^{-1}]$ such that

$$\psi_\varepsilon(s) := [G''_\varepsilon(s)]^{-1} := \begin{cases} \varepsilon & \text{if } s \leq \varepsilon\,, \\ s & \text{if } \varepsilon \leq s \leq \varepsilon^{-1}\,, \\ \frac{1}{\varepsilon} & \text{if } s \geq \varepsilon^{-1}\,. \end{cases} \tag{2.3.11}$$

Similarly to the regularized functions $F_\varepsilon(s)$ and $\phi_\varepsilon(s)$, on noting (2.3.10a)-(2.3.10c) and (2.3.11), it is easy to show that the following properties concerning the functions $G_\varepsilon(s)$ and $\psi_\varepsilon(s)$ hold for all $\varepsilon \in (0, e^{-1})$:

$$G_\varepsilon(s) \geq \tfrac{s^2}{2\,\varepsilon} \qquad \text{for all } s \leq 0, \tag{2.3.12}$$

$$G_\varepsilon(s) \geq \tfrac{\varepsilon\, s^2}{2} - 1 \qquad \text{for all } s \geq 0, \tag{2.3.13}$$

$$\max\{\psi_\varepsilon(s),\; s\, G_\varepsilon'(s)\} \leq 2\, G_\varepsilon(s) + 1 \qquad \text{for all } s \in \mathbb{R}, \tag{2.3.14}$$

$$\psi_\varepsilon(s)\, G_\varepsilon'(s) \geq s - 1 \qquad \text{for all } s \in \mathbb{R}. \tag{2.3.15}$$

We now consider the corresponding regularized version of the problem (P_M) for any $\varepsilon \in (0, e^{-1})$:

($\mathbf{P}_{M,\varepsilon}$) For fixed $M \geq e$, find $\{u_\varepsilon(x,t), v_\varepsilon(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\frac{\partial u_\varepsilon}{\partial t} = \nabla \cdot (\, D\, \nabla u_\varepsilon + \phi_\varepsilon(u_\varepsilon)\, \nabla(u_\varepsilon + v_\varepsilon)) + f_{M,\varepsilon}(u_\varepsilon, v_\varepsilon) \qquad \text{in } \Omega_T, \tag{2.3.16a}$$

$$\frac{\partial v_\varepsilon}{\partial t} = \nabla \cdot (\, D\, \nabla v_\varepsilon + \psi_\varepsilon(v_\varepsilon)\, \nabla(u_\varepsilon + v_\varepsilon)) + g_{M,\varepsilon}(u_\varepsilon, v_\varepsilon) \qquad \text{in } \Omega_T, \tag{2.3.16b}$$

with boundary conditions

$$\left.\begin{aligned} &[\, D\, \nabla u_\varepsilon + \phi_\varepsilon(u_\varepsilon)\, \nabla(u_\varepsilon + v_\varepsilon)] \cdot \nu = 0 \\ &[\, D\, \nabla v_\varepsilon + \psi_\varepsilon(v_\varepsilon)\, \nabla(u_\varepsilon + v_\varepsilon)] \cdot \nu = 0 \end{aligned}\right. \qquad \text{on } \partial\Omega \times (0,T), \tag{2.3.16c}$$

and initial conditions

$$u_\varepsilon(x,0) = u^0(x)\,, \quad v_\varepsilon(x,0) = v^0(x) \qquad \forall\, x \in \Omega; \tag{2.3.16d}$$

where

$$f_{M,\varepsilon}(u_\varepsilon, v_\varepsilon) := u_\varepsilon - \phi_\varepsilon(u_\varepsilon)\,(\, u_\varepsilon + \psi_\varepsilon(v_\varepsilon)\,),$$

$$g_{M,\varepsilon}(u_\varepsilon, v_\varepsilon) := \gamma\, v_\varepsilon - \psi_\varepsilon(v_\varepsilon)\,(\, \phi_\varepsilon(u_\varepsilon) + \psi_\varepsilon(v_\varepsilon)\,).$$

Here, the functions $f_{M,\varepsilon}$ and $g_{M,\varepsilon}$ are considered to be appropriate to control the non-linearity and obtain the intended results. Later, we discuss other possible choices of $f_{M,\varepsilon}$ and $g_{M,\varepsilon}$ (see Remark 3.3.1 and Remark 3.3.2).

In the following lemma we derive an analogue to the entropy inequality (2.3.3) for the regularized problem ($\mathrm{P}_{M,\varepsilon}$) which will provide us with some uniform bounds on the regularized solutions u_ε and v_ε under our assumption that $D > 0$.

Lemma 2.3.1 Let $u^0(x)$ and $v^0(x)$ be non-negative bounded functions. There exists a positive $C(u^0, v^0, M, \gamma)$ independent of ε such that any solution of $(\mathrm{P}_{M,\varepsilon})$ satisfies

$$\sup_{0<t<T} \int_\Omega \Big(F_\varepsilon(u_\varepsilon) + G_\varepsilon(v_\varepsilon) \Big) \, \mathrm{d}x + \tfrac{D}{M} \int_{\Omega_T} |\nabla u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \\ + \int_{\Omega_T} |\nabla u_\varepsilon + \nabla v_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C\,. \tag{2.3.17}$$

In addition,

$$\sup_{0<t<T} \int_\Omega \Big(|[u_\varepsilon]_-|^2 + |[v_\varepsilon]_-|^2 \Big) \, \mathrm{d}x \leq C\, \varepsilon. \tag{2.3.18}$$

Proof: Multiplying (2.3.16a) and (2.3.16b) by $F'_\varepsilon(u_\varepsilon)$ and $G'_\varepsilon(v_\varepsilon)$ respectively, integrating by parts over the domain Ω, summing the resulting equations and noting (2.3.5) and (2.3.11) yields, after recalling the boundary conditions (2.3.16c), that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Big(F_\varepsilon(u_\varepsilon) + G_\varepsilon(v_\varepsilon) \Big) \, \mathrm{d}x + D \int_\Omega \left(\frac{|\nabla u_\varepsilon|^2}{\phi_\varepsilon(u_\varepsilon)} + \frac{|\nabla v_\varepsilon|^2}{\psi_\varepsilon(v_\varepsilon)} \right) \mathrm{d}x + \int_\Omega |\nabla u_\varepsilon + \nabla v_\varepsilon|^2 \, \mathrm{d}x \\ = \int_\Omega \Big(f_{M,\varepsilon}(u_\varepsilon, v_\varepsilon) \, F'_\varepsilon(u_\varepsilon) + g_{M,\varepsilon}(u_\varepsilon, v_\varepsilon) \, G'_\varepsilon(v_\varepsilon) \Big) \, \mathrm{d}x\,, \tag{2.3.19}$$

where we have noticed that

$$\phi_\varepsilon(u_\varepsilon) \nabla \left[F'_\varepsilon(u_\varepsilon) \right] = \nabla u_\varepsilon\,, \tag{2.3.20}$$

$$\psi_\varepsilon(v_\varepsilon) \nabla \left[G'_\varepsilon(v_\varepsilon) \right] = \nabla v_\varepsilon\,. \tag{2.3.21}$$

It follows from (2.3.5), (2.3.11), (2.3.8), (2.3.9), (2.1.11), (2.3.14), the Young's inequality and (2.3.6) that

$$\begin{aligned} f_{M,\varepsilon}(u_\varepsilon, v_\varepsilon) \, F'_\varepsilon(u_\varepsilon) &= u_\varepsilon \, F'_\varepsilon(u_\varepsilon) - \phi_\varepsilon(u_\varepsilon) \, (u_\varepsilon \, F'_\varepsilon(u_\varepsilon)) - \psi_\varepsilon(v_\varepsilon) \, (\phi_\varepsilon(u_\varepsilon) \, F'_\varepsilon(u_\varepsilon)) \\ &\leq (2 \, F_\varepsilon(u_\varepsilon) + 1) + (1 - u_\varepsilon) \, (\phi_\varepsilon(u_\varepsilon) + \psi_\varepsilon(v_\varepsilon)) \\ &\leq (2 \, F_\varepsilon(u_\varepsilon) + 1) + (1 - [u_\varepsilon]_-) \, (\phi_\varepsilon(u_\varepsilon) + \psi_\varepsilon(v_\varepsilon)) \\ &\leq 2 \, F_\varepsilon(u_\varepsilon) + 2 \, G_\varepsilon(v_\varepsilon) + \tfrac{1}{\varepsilon} \, [u_\varepsilon]_-^2 + \tfrac{\varepsilon}{2} \, (\phi_\varepsilon^2(u_\varepsilon) + \psi_\varepsilon^2(v_\varepsilon)) + C(M) \\ &\leq 4 \, F_\varepsilon(u_\varepsilon) + 2 \, G_\varepsilon(v_\varepsilon) + \tfrac{1}{2} \, \psi_\varepsilon(v_\varepsilon) + C(M) \\ &\leq 4 \, F_\varepsilon(u_\varepsilon) + 3 \, G_\varepsilon(v_\varepsilon) + C(M)\,. \end{aligned} \tag{2.3.22}$$

Similarly to (2.3.22), on noting (2.3.5), (2.3.11), (2.3.14), (2.3.15), (2.1.11), the Young's inequality and (2.3.12), we have that

$$
\begin{aligned}
g_{M,\varepsilon}(u_\varepsilon, v_\varepsilon)\, G'_\varepsilon(v_\varepsilon) &= \gamma\, v_\varepsilon\, G'_\varepsilon(v_\varepsilon) - \psi_\varepsilon(v_\varepsilon)\, G'_\varepsilon(v_\varepsilon)\ \left(\phi_\varepsilon(u_\varepsilon) + \psi_\varepsilon(v_\varepsilon)\right) \\
&\le \gamma\ \left(2\, G_\varepsilon(v_\varepsilon) + 1\right) + (1 - v_\varepsilon)\ \left(\phi_\varepsilon(u_\varepsilon) + \psi_\varepsilon(v_\varepsilon)\right) \\
&\le \gamma\ \left(2\, G_\varepsilon(v_\varepsilon) + 1\right) + (1 - [v_\varepsilon]_-)\ \left(\phi_\varepsilon(u_\varepsilon) + \psi_\varepsilon(v_\varepsilon)\right) \\
&\le (2\,\gamma + 2)\ G_\varepsilon(v_\varepsilon) + \tfrac{1}{\varepsilon}\, [v_\varepsilon]_-^2 + \tfrac{\varepsilon}{2}\ \left(\phi^2_\varepsilon(u_\varepsilon) + \psi^2_\varepsilon(v_\varepsilon)\right) + C(M,\gamma) \\
&\le (2\,\gamma + 4)\ G_\varepsilon(v_\varepsilon) + \tfrac{1}{2}\, \psi_\varepsilon(v_\varepsilon) + C(M,\gamma) \\
&\le (2\,\gamma + 5)\ G_\varepsilon(v_\varepsilon) + C(M,\gamma)\,.
\end{aligned} \tag{2.3.23}
$$

Combining (2.3.19), (2.3.22) and (2.3.23) yields, on noting (2.3.5) and (2.3.11), that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Big(F_\varepsilon(u_\varepsilon) + G_\varepsilon(v_\varepsilon) \Big)\, \mathrm{d}x + \tfrac{D}{M} \int_\Omega |\nabla u_\varepsilon|^2\, \mathrm{d}x + \int_\Omega |\nabla u_\varepsilon + \nabla v_\varepsilon|^2\, \mathrm{d}x \\
\le C(M,\gamma)\ \left(1 + \int_\Omega \Big(F_\varepsilon(u_\varepsilon) + G_\varepsilon(v_\varepsilon) \Big)\, \mathrm{d}x \right).
\end{aligned} \tag{2.3.24}
$$

Applying the Grönwall inequality (2.1.6), on recalling the initial conditions (2.3.16d) and the assumption on u^0 and v^0, leads to the desired result (2.3.17). The result (2.3.18) follows immediately from (2.3.17), (2.3.6) and (2.3.12). □

On noting (2.3.17), the triangle inequality and the Poincaré inequality, one can obtain a uniform $L^2(0,T;H^1(\Omega))$ bound on the solutions u_ε and v_ε independently of the regularization parameter ε. Furthermore, a uniform $L^\infty(0,T;L^2(\Omega))$ bound on u_ε can be easily obtained from (2.3.17), (2.3.6) and (2.3.7).

The existence of a non-negative solution of (P_M) can be shown by passing to the limit $\varepsilon \to 0$ on noting (2.3.17) and (2.3.18) where the estimate (2.3.18) is the key to prove the non-negativity of the solution. However, this can only be performed in the case that we have existence of a solution to the regularized problem $(\mathrm{P}_{M,\varepsilon})$. To deal with this issue, in our study of problem (P_M), we use the power of the finite element method.

We now formulate a fully discrete finite element approximation of $(\mathrm{P}_{M,\varepsilon})$ and prove existence of fully discrete approximation solutions.

2.4 A fully discrete finite element approximation

In this section we introduce a practical fully discrete finite element approximation of the system $(\mathrm{P}_{M,\varepsilon})$. To do that we discretise the system in space using the finite element method and discretise in time using the finite differences. In Subsection 2.4.1 we recall definitions of different types of partitioning in space. We state the required assumptions on the partitioning of Ω and $(0,T)$. We also define the standard finite element space and discuss some associated results. In Subsection 2.4.2 we formulate a practical fully discrete finite element approximation of the system $(\mathrm{P}_{M,\varepsilon})$ and prove some technical lemmata. Then, in Subsection 2.4.3, we prove existence of the finite element approximations under appropriate assumption on the discretization parameters.

2.4.1 Notation and associated results

Let $\mathcal{T}^h$ be a partitioning of $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. A simplex $\tau \in \mathcal{T}^h$ is defined as an interval if $d = 1$, a triangle if $d = 2$ and a tetrahedron if $d = 3$. We define $h_\tau := \operatorname{diam} \tau$ to be the length of τ if $d = 1$, the longest side of τ if $d = 2$ and the longest edge of τ if $d = 3$. The parameter h indicates the maximal diameter of the simplices of the partitioning. We recall that a partitioning $\mathcal{T}^h$ is said to be "quasi-uniform" if there exists a positive constant β such that

$$\frac{\varrho_\tau}{h_\tau} \;\geq\; \beta \qquad\qquad \forall\, \tau \in \mathcal{T}_h,$$

where ϱ_τ denotes the diameter of the sphere inscribed in τ. For instance, in the case $d = 2$, the quasi-uniform condition means that the angles of the triangles $\tau \in \mathcal{T}^h$ are not allowed to be arbitrarily small; see Johnson [42] page 85. We also recall that a partitioning $\mathcal{T}^h$ is said to be "acute" for $d = 2$ if all the angles of the triangles are less than or equal to $\pi/2$, and for $d = 3$ if the angles made by any two faces of the same tetrahedron are less than or equal to $\pi/2$. Another type of partitioning is the "right-angled" that is, in the case $d = 2$, if all triangles are right-angled; and in the case $d = 3$, if all tetrahedra have a vertex at which all the edges meet at right angles. From the definitions, we note that the right-angled partitioning is acute.

In the work that follows we consider the finite element approximation of $(\mathrm{P}_{M,\varepsilon})$ under the following assumptions on the spacies and temporal meshes:

(A) Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a polygonal domain in $d = 2$ and a polyhedral domain in $d = 3$. Let $\mathcal{T}^h$ be a quasi-uniform and right-angled partitioning of Ω into disjoint open simplices $\{\,\tau\,\}$ with $h_\tau := \operatorname{diam}\tau$ and $h := \max_{\tau\in\mathcal{T}^h} h_\tau$, so that $\overline{\Omega} = \bigcup_{\tau\in\mathcal{T}^h} \overline{\tau}$. Let $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$ be a partitioning of $(0, T)$ into time steps $\Delta t_n := t_n - t_{n-1}$, $n = 1, \cdots, N$, with $\Delta t := \max_{n=1,\cdots,N} \Delta t_n$.

Let $S^h \subset H^1(\Omega)$ be the standard finite element space of continuous piecewise linear function:

$$S^h := \{\chi \in C(\overline{\Omega}) : \chi\,|_\tau \text{ is linear } \forall\, \tau \in \mathcal{T}^h\}.$$

Denote by $\mathcal{N}^h := \{\,p_j\,\}_{j=0}^J$ the set of nodes of the partitioning $\mathcal{T}^h$ and let $\{\,\varphi_j\,\}_{j=0}^J$ be the canonical basis functions associated with S^h, satisfying $\varphi_j(p_i) = \delta_{ij}$ for $i, j = 0, \cdots, J$.

We also introduce

$$\begin{aligned} S^h_{\geq 0} &:= \{\chi \in S^h : \chi(p_j) \geq 0,\ j = 0, \cdots, J\} \\ &\subset H^1_{\geq 0}(\Omega) := \{\eta \in H^1(\Omega) : \eta \geq 0 \ a.e. \text{ in } \Omega\}\,. \end{aligned}$$

Let $\pi^h : C(\overline{\Omega}) \to S^h$ be the interpolation operator such that for all $\eta \in C(\overline{\Omega})$

$$\pi^h\eta(p_j) := \eta(p_j) \qquad \text{for } j = 0, \cdots, J,$$

and define a discrete semi-inner product on $C(\overline{\Omega})$, and inner product on S^h, as follows

$$(u, v)^h := \int_\Omega \pi^h(u(x)\,v(x))\,\mathrm{d}x = \sum_{j=0}^J \widehat{M}_{jj}\,u(p_j)\,v(p_j), \tag{2.4.1}$$

where $\widehat{M}_{jj} := (\varphi_j, \varphi_j)^h = (1, \varphi_j) > 0$. The induced discrete semi-norm on $C(\overline{\Omega})$, and norm on S^h, is $|\,.\,|_h := [\,(\,.\,,\,.\,)^h\,]^{1/2}$. We note that $|\,.\,|_h$ is equivalent to the norm $||\,.\,||_0 := [\,(\,.\,,\,.\,)\,]^{1/2}$. Namely,

$$||\chi||_0^2 \leq |\chi|_h^2 \leq (d+2)\,||\chi||_0^2 \qquad \forall \chi \in S^h, \tag{2.4.2}$$

(see, e.g., Raviart [56]).

On noting (2.4.1) it is easy to show that

$$(\eta_1, \eta_2)^h = (\pi^h \eta_1, \eta_2)^h = (\pi^h \eta_1, \pi^h \eta_2)^h \qquad \forall \eta_1, \eta_2 \in C(\overline{\Omega}). \tag{2.4.3}$$

Let $\widehat{M} := \left(\widehat{M}_{ij}\right)_{i,j=0}^{J}$ and $K := \left(K_{ij}\right)_{i,j=0}^{J}$ to be the lumped mass matrix and the stiffness matrix, respectively, where

$$\widehat{M}_{ij} := (\varphi_i, \varphi_j)^h \qquad \text{and} \qquad K_{ij} := (\nabla \varphi_i, \nabla \varphi_j)\,.$$

As the partitioning $\mathcal{T}^h$ is acute, we have that (see [53] page 49)

$$K_{jj} > 0 \;\; \forall\, j \qquad \text{and} \qquad K_{ij} \leq 0 \;\; \forall\, i \neq j\,. \tag{2.4.4}$$

Using the fact $\sum\limits_{j=0}^{J} \varphi_j = 1$, we also have

$$\sum_{j=0}^{J} K_{ij} = \left(\nabla \varphi_i, \nabla \sum_{j=0}^{J} \varphi_j\right) = 0\,. \tag{2.4.5}$$

Providing that the partitioning $\mathcal{T}^h$ is acute, we prove the following lemma about the regularized functions $\phi_\varepsilon(s)$ and $\psi_\varepsilon(s)$ which will be needed later on to derive some useful estimates. See Section 2.4.2 in [53] and Section 4.2 in [35] for similar results.

Lemma 2.4.1 Let the assumptions (A) hold. Then for all $\chi \in S^h$

$$||\nabla \pi^h[\phi_\varepsilon(\chi)]||_0^2 \leq (\nabla \chi, \nabla \pi^h[\phi_\varepsilon(\chi)])\,, \tag{2.4.6}$$

$$||\nabla \pi^h[\psi_\varepsilon(\chi)]||_0^2 \leq (\nabla \chi, \nabla \pi^h[\psi_\varepsilon(\chi)])\,. \tag{2.4.7}$$

Proof: We prove (2.4.6) and the proof of (2.4.7) will follow similarly. On noting (2.3.5) we find that

$$(\phi_\varepsilon(a) - \phi_\varepsilon(b))^2 \leq (\phi_\varepsilon(a) - \phi_\varepsilon(b))\;(a-b) \qquad \forall\, a, b \in \mathbb{R}\,, \tag{2.4.8}$$

where we have noticed that ϕ_ε is 1-Lipschitz continuous and non-decreasing function. Also, as $K_{ij} = K_{ji}$, we note for any $a_j,\; b_j,\; c_j \in \mathbb{R}$ that

$$\sum_{i=0}^{J} \sum_{\substack{j=0\\ j\neq i}}^{J} a_i\;(b_j - c_i)\;K_{ij} = \sum_{i=0}^{J} \sum_{\substack{j=0\\ j\neq i}}^{J} a_j\;(b_i - c_j)\;K_{ij}\,. \tag{2.4.9}$$

A function $\chi \in S^h$ can be expressed as $\chi := \sum_{j=0}^{J} \chi_j \, \varphi_j$ where $\chi_j := \chi(p_j)$, $j = 0, \cdots, J$. Noting this and (2.4.5) we have that

$$
\begin{aligned}
(\nabla\chi, \nabla\pi^h[\phi_\varepsilon(\chi)]) &= \left(\nabla \sum_{i=0}^{J} \chi_i \, \varphi_i \, , \nabla \sum_{j=0}^{J} \phi_\varepsilon(\chi_j) \, \varphi_j \right) \\
&= \sum_{i=0}^{J} \sum_{j=0}^{J} \chi_i \, \phi_\varepsilon(\chi_j) \, K_{ij} \\
&= \sum_{i=0}^{J} \sum_{\substack{j=0 \\ j\neq i}}^{J} \chi_i \, \phi_\varepsilon(\chi_j) \, K_{ij} + \sum_{i=0}^{J} \chi_i \, \phi_\varepsilon(\chi_i) \, K_{ii} \\
&= \sum_{i=0}^{J} \sum_{\substack{j=0 \\ j\neq i}}^{J} \chi_i \, \phi_\varepsilon(\chi_j) \, K_{ij} - \sum_{i=0}^{J} \sum_{\substack{j=0 \\ j\neq i}}^{J} \chi_i \, \phi_\varepsilon(\chi_i) \, K_{ij} \\
&= \sum_{i=0}^{J} \sum_{\substack{j=0 \\ j\neq i}}^{J} \chi_i \, \left(\phi_\varepsilon(\chi_j) - \phi_\varepsilon(\chi_i)\right) \, K_{ij} \, .
\end{aligned}
\tag{2.4.10}
$$

It follows from (2.4.10) and the notation (2.4.9) that

$$
\begin{aligned}
(\nabla\chi, \nabla\pi^h[\phi_\varepsilon(\chi)]) &= \sum_{i=0}^{J} \sum_{\substack{j=0 \\ j\neq i}}^{J} \chi_i \, \left(\phi_\varepsilon(\chi_j) - \phi_\varepsilon(\chi_i)\right) \, K_{ij} \\
&= \frac{1}{2} \sum_{i=0}^{J} \sum_{\substack{j=0 \\ j\neq i}}^{J} \Big[\chi_i \, \left(\phi_\varepsilon(\chi_j) - \phi_\varepsilon(\chi_i)\right) \, K_{ij} + \chi_j \, \left(\phi_\varepsilon(\chi_i) - \phi_\varepsilon(\chi_j)\right) \, K_{ij}\Big] \\
&= \frac{1}{2} \sum_{i=0}^{J} \sum_{\substack{j=0 \\ j\neq i}}^{J} (-K_{ij}) \, (\chi_i - \chi_j) \, \left(\phi_\varepsilon(\chi_i) - \phi_\varepsilon(\chi_j)\right) \, .
\end{aligned}
\tag{2.4.11}
$$

Similarly to (2.4.11), we obtain that

$$
(\nabla\pi^h[\phi_\varepsilon(\chi)], \nabla\pi^h[\phi_\varepsilon(\chi)]) = \frac{1}{2} \sum_{i=0}^{J} \sum_{\substack{j=0 \\ j\neq i}}^{J} (-K_{ij}) \, \left(\phi_\varepsilon(\chi_i) - \phi_\varepsilon(\chi_j)\right)^2 \, . \tag{2.4.12}
$$

Combining (2.4.11), (2.4.12), (2.4.4) and (2.4.8) yields the desired inequality (2.4.6). The result (2.4.7) can be shown by following the same argument used for (2.4.6) on noting that ψ_ε is also 1-Lipschitz continuous and non-decreasing. $\square$

We now recall some well-known results about the space S^h under our assumption that $\mathcal{T}^h$ is quasi-uniform partitioning:

For any $\tau \in \mathcal{T}^h$, $\chi \in S^h$, $1 \le p\,,\, q \le \infty$ and $m,\ l \in \{0,\ 1\}$ with $l \le m$, we have

$$||\chi||_{m,p,\text{``}\tau\text{''}} \le C\, h_{\text{``}\tau\text{''}}^{(l-m)\,+\,d\,\min(0,\ \frac{1}{p}-\frac{1}{q})}\, ||\chi||_{l,q,\text{``}\tau\text{''}}\,, \qquad (2.4.13)$$

where the abbreviation "τ" means "with" or "without" τ, i.e. with τ or with Ω. The above inequality is known as "the inverse inequality", see [32] page 75-77, and it also holds with $||\,.\,||$ replaced by $|\,.\,|$, see [23] page 140-142.

In particular, in our work, we will make frequent use of the following cases of the inverse inequality

$$|\chi|_{1,p,\text{``}\tau\text{''}} \le C\, h_{\text{``}\tau\text{''}}^{-1}\, |\chi|_{0,p,\text{``}\tau\text{''}} \qquad 1 \le p \le \infty\,, \qquad (2.4.14)$$

$$|\chi|_{m,p,\text{``}\tau\text{''}} \le C\, h_{\text{``}\tau\text{''}}^{-d\,(\frac{1}{q}-\frac{1}{p})}\, |\chi|_{m,q,\text{``}\tau\text{''}} \qquad 1 \le q \le p \le \infty,\ m \in \{0,\ 1\}\,. \qquad (2.4.15)$$

We also require the following interpolation results for all $\eta \in W^{1,s}(\Omega)\,,\, s \in [2, \infty]$ if $d = 1$ and $s \in (d, \infty]$ if $d = 2$ or 3:

$$|(I - \pi^h)\eta|_{m,s} \le C\, h^{1-m}\, |\eta|_{1,s} \qquad m \in \{0,\ 1\}\,, \qquad (2.4.16)$$

$$\lim_{h\to 0} \|(I - \pi^h)\eta\|_{1,s} = 0\,, \qquad (2.4.17)$$

(see Theorem 1.103 and Corollary 1.110 in [32] respectively).

Due to the quasi-uniform partitioning of $\mathcal{T}^h$, we have for all $\eta \in W^{2,1}(\Omega)$ that (see Theorem 5 in [24]):

$$||(I - \pi^h)\eta||_{0,1} \le C\, h^2\, |\eta|_{2,1}. \qquad (2.4.18)$$

It is easily established, see for instance the proof of Lemma 2.4.2 , from (2.4.1), (2.4.18), the Hölder's inequality and (2.4.14), for all $\chi_1,\ \chi_2 \in S^h$, that

$$\begin{aligned} \left|(\chi_1, \chi_2) - (\chi_1, \chi_2)^h\right| &\le \left\|(I - \pi^h)(\chi_1\, \chi_2)\right\|_{0,1} \\ &\le C\, h^{1+m}\, |\chi_1|_{m,n_1}\, |\chi_2|_{1,n_2}\,, \end{aligned} \qquad (2.4.19)$$

for $m \in \{0,\ 1\}$ and $1 \le n_1,\ n_2 \le \infty$ with $\frac{1}{n_1} + \frac{1}{n_2} = 1\,.$

For later purposes, we prove the following generalized version of the estimate (2.4.19).

Lemma 2.4.2 For all $\chi_1\,,\ \chi_2\,,\ \chi_3 \in S^h$

$$\left|(\chi_1\,\chi_2,\chi_3) - (\chi_1\,\chi_2,\chi_3)^h\right| \leq C\,h^2\,||\chi_1||_{1,n_1}\,||\chi_2||_{1,n_2}\,||\chi_2||_{1,n_3}\,, \tag{2.4.20}$$

where $\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}=1\,,\ \ 1\leq n_1,\ n_2,\ n_3\leq\infty.$

Proof: On noting the generalized Hölder's inequality we have for $k_1, k_2, k_3 = 1,2,3$, for $i, j = 1, \cdots, d$ and for any $\tau \in \mathcal{T}^h$ that

$$\left\|\frac{\partial\chi_{k_1}}{\partial x_i}\,\frac{\partial\chi_{k_2}}{\partial x_j}\,\chi_{k_3}\right\|_{0,1,\tau} \leq ||\chi_{k_1}||_{1,n_1,\tau}\,||\chi_{k_2}||_{1,n_2,\tau}\,||\chi_{k_3}||_{1,n_3,\tau}\,, \tag{2.4.21}$$

where $\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}=1\,,\ \ 1\leq n_1,\ n_2,\ n_3\leq\infty.$
We now have from the definition (2.4.1), (2.4.18), (2.4.21) and the generalized discrete Hölder's inequality that

$$\begin{aligned}
\left|(\chi_1\,\chi_2,\chi_3) - (\chi_1\,\chi_2,\chi_3)^h\right| &= \left|\int_\Omega (I-\pi^h)(\chi_1\,\chi_2\,\chi_3)\ \ \mathrm{d}x\right| \\
&\leq C\,h^2\sum_{\tau\in\mathcal{T}^h}\sum_{|\alpha|=2}\left\|D^\alpha(\chi_1\,\chi_2\,\chi_3)\right\|_{0,1,\tau} \\
&\leq C\,h^2\sum_{\tau\in\mathcal{T}^h}\sum_{i,j=1}^{d}\left\|\frac{\partial^2(\chi_1\,\chi_2\,\chi_3)}{\partial x_i\,\partial x_j}\right\|_{0,1,\tau} \\
&\leq C\,h^2\sum_{\tau\in\mathcal{T}^h}\sum_{i,j=1}^{d}\sum_{\substack{k_1,k_2,k_3=1\\ k_1\neq k_2\neq k_3\neq k_1}}^{3}\left\|\frac{\partial\chi_{k_1}}{\partial x_i}\,\frac{\partial\chi_{k_2}}{\partial x_j}\,\chi_{k_3}\right\|_{0,1,\tau} \\
&\leq C\,h^2\sum_{\tau\in\mathcal{T}^h}\sum_{i,j=1}^{d}\left[6\,||\chi_1||_{1,n_1,\tau}\,||\chi_2||_{1,n_2,\tau}\,||\chi_3||_{1,n_3,\tau}\right] \\
&\leq C\,h^2\sum_{\tau\in\mathcal{T}^h}||\chi_1||_{1,n_1,\tau}\,||\chi_2||_{1,n_2,\tau}\,||\chi_3||_{1,n_3,\tau} \\
&\leq C\,h^2\left(\sum_{\tau\in\mathcal{T}^h}||\chi_1||_{1,n_1,\tau}^{n_1}\right)^{\frac{1}{n_1}}\left(\sum_{\tau\in\mathcal{T}^h}||\chi_2||_{1,n_2,\tau}^{n_2}\right)^{\frac{1}{n_2}}\left(\sum_{\tau\in\mathcal{T}^h}||\chi_3||_{1,n_3,\tau}^{n_3}\right)^{\frac{1}{n_3}} \\
&\leq C\,h^2\,||\chi_1||_{1,n_1}\,||\chi_2||_{1,n_2}\,||\chi_2||_{1,n_3}\,.
\end{aligned}$$

□

We are now in a position to formulate a practical fully discrete finite element approximation of the system $(\mathrm{P}_{M,\varepsilon})$.

2.4.2 A practical fully discrete approximation

In order to introduce a fully discrete approximation that is consistent with the regularized problem $(\mathrm{P}_{M,\varepsilon})$, we adapt a technique developed in [36] for studying a degenerate nonlinear fourth order parabolic equation modelling the height of thin films of viscous fluids driven by surface tension. This technique has been also adapted and employed in a number of numerical studies, see for example [9], [10], [11], [12] and [13].

We define, for any $\varepsilon \in (0, e^{-1})$, a function $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ such that for all $\chi \in S^h$ and *a.e.* in Ω

$$\Lambda_\varepsilon(\chi) \text{ is symmetric and positive definite,} \tag{2.4.22a}$$

$$\Lambda_\varepsilon(\chi)\,\nabla\pi^h[F'_\varepsilon(\chi)] = \nabla\chi; \tag{2.4.22b}$$

that is, the discrete analogue to (2.3.20). In the next few lines, we follow Grün *et al.* [36] to give the construction of Λ_ε on each simplex $\tau \in \mathcal{T}^h$ for any given $\chi \in S^h$. In one space dimension, we set

$$\Lambda_\varepsilon(\chi)\,|_\tau := \begin{cases} \frac{\chi(p_k)-\chi(p_j)}{F'_\varepsilon(\chi(p_k))-F'_\varepsilon(\chi(p_j))} = \frac{1}{F''_\varepsilon(\chi(\zeta))} & \\ \qquad\qquad \text{for some } \zeta\in\tau & \text{if } \chi(p_k)\neq\chi(p_j), \\ \frac{1}{F''_\varepsilon(\chi(p_k))} & \text{if } \chi(p_k)=\chi(p_j), \end{cases} \tag{2.4.23}$$

where p_j and p_k are the vertices of the interval τ. Since $F''_\varepsilon(s) > 0$ and $\sum_{j=0}^{J} \nabla\varphi_j = 0$, it can be easily seen that the piecewise constant function Λ_ε satisfies the conditions (2.4.22a,b).

We now consider the case when $d = 2$ or 3. Let $\{e_i\}_{i=1}^d$ be the orthonormal vectors in $\mathbb{R}^d$, such that e_i denotes the i-th unit vector. Given non-zero constants α_i, $i = 1, \cdots, d$, we define $\widehat{\tau}(\{\alpha_i\}_{i=1}^d)$ to be a reference open simplex in $\mathbb{R}^d$ with vertices $\{\widehat{p}_i\}_{i=0}^d$, where $\widehat{p}_0$ is the origin and $\widehat{p}_i := \alpha_i\, e_i$, $i = 1, \cdots, d$. Note that the simplex $\widehat{\tau}$ is right angled in the vertex $\widehat{p}_0$. Using this notation, we now define the function Λ_ε on each element of $\mathcal{T}^h$. On recalling assumption (A) that the partitioning $\mathcal{T}^h$ is right-angled, let $\tau \in \mathcal{T}^h$ with vertices $\{p_{j_i}\}_{i=0}^d$, such that p_{j_0} is a right angled vertex. We can find non-zero constants $\{\alpha_i\}_{i=1}^d$ and an orthogonal matrix R_τ in such a way that the mapping $\mathcal{R}_\tau : \widehat{x} \in \mathbb{R}^d \to p_{j_0} + R_\tau \widehat{x} \in \mathbb{R}^d$ maps the vertex $\widehat{p}_i$ to p_{j_i}, $i = 0, \cdots, d$, and hence $\widehat{\tau} \equiv \widehat{\tau}(\{\alpha_i\}_{i=1}^d)$ to τ. For any $\tau \in \mathcal{T}^h$ and $\chi \in S^h$, we then set

$$\Lambda_\varepsilon(\chi)\,|_\tau := R_\tau\, \widehat{\Lambda}_\varepsilon(\widehat{\chi})\ |_{\widehat{\tau}}\, R_\tau^T, \tag{2.4.24}$$

where $\widehat{\chi}(\widehat{x}) \equiv \chi(\mathcal{R}_\tau \widehat{x})$ for all $\widehat{x} \in \overline{\widehat{\tau}}$ and $\widehat{\Lambda}_\varepsilon(\widehat{\chi})\,|_{\widehat{\tau}}$ is the $d \times d$ diagonal matrix with diagonal entries, $k = 1, \cdots, d$,

$$[\widehat{\Lambda}_\varepsilon(\widehat{\chi})\,|_{\widehat{\tau}}]_{kk} := \begin{cases} \frac{\widehat{\chi}(\widehat{p}_k)-\widehat{\chi}(\widehat{p}_0)}{F'_\varepsilon(\widehat{\chi}(\widehat{p}_k))-F'_\varepsilon(\widehat{\chi}(\widehat{p}_0))} \equiv \frac{\chi(p_{j_k})-\chi(p_{j_0})}{F'_\varepsilon(\chi(p_{j_k}))-F'_\varepsilon(\chi(p_{j_0}))} & \\ \qquad = \frac{1}{F''_\varepsilon(\chi(\zeta))} \quad \text{for some } \zeta \text{ between } p_{j_k} \text{ and } p_{j_0} & \\ \qquad\qquad \text{if} \quad \chi(p_{j_k}) \neq \chi(p_{j_0}), & \\ \frac{1}{F''_\varepsilon(\widehat{\chi}(\widehat{p}_0))} \equiv \frac{1}{F''_\varepsilon(\chi(p_{j_0}))} \qquad \text{if} \quad \chi(p_{j_k}) = \chi(p_{j_0}). & \end{cases} \tag{2.4.25}$$

As $R_\tau^T \equiv R_\tau^{-1}$, we have that

$$\nabla \chi\,|_\tau \equiv R_\tau\, \widehat{\nabla} \widehat{\chi}\,|_{\widehat{\tau}}\,, \tag{2.4.26}$$

where $\widehat{\nabla}$ is the gradient on $\widehat{\tau}$. On noting (2.4.24), (2.4.25), (2.4.26), the positivity of $F''_\varepsilon(s)$ and the fact $\sum_{j=0}^{J} \nabla \varphi_j = 0$, one can easily show that Λ_ε satisfies the conditions (2.4.22a,b).

In a similar fashion, for any $\varepsilon \in (0, e^{-1})$, we introduce a function $\Xi_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ such that for all $\chi \in S^h$ and $a.e.$ in Ω

$$\Xi_\varepsilon(\chi) \text{ is symmetric and positive definite,} \tag{2.4.27a}$$

$$\Xi_\varepsilon(\chi)\, \nabla \pi^h[G'_\varepsilon(\chi)] = \nabla \chi; \tag{2.4.27b}$$

that is, the discrete analogue to (2.3.21). The construction (2.4.23)-(2.4.25) for Λ_ε can be extended to Ξ_ε. In the case $d = 1$, we set for any $\chi \in S^h$ and $\tau \in \mathcal{T}^h$ having the vertices p_j and p_k,

$$\Xi_\varepsilon(\chi)\,|_\tau := \begin{cases} \frac{\chi(p_k)-\chi(p_j)}{G'_\varepsilon(\chi(p_k))-G'_\varepsilon(\chi(p_j))} = \frac{1}{G''_\varepsilon(\chi(\zeta))} & \\ \qquad \text{for some } \zeta \in \tau \quad \text{if} \quad \chi(p_k) \neq \chi(p_j), & \\ \frac{1}{G''_\varepsilon(\chi(p_k))} \qquad \text{if} \quad \chi(p_k) = \chi(p_j). & \end{cases} \tag{2.4.28}$$

When $d = 2$ or 3, we set

$$\Xi_\varepsilon(\chi)\,|_\tau := R_\tau\,\widehat{\Xi}_\varepsilon(\widehat{\chi})\;\;|_{\widehat{\tau}}\;R_\tau^T, \tag{2.4.29}$$

where $\widehat{\Xi}_\varepsilon(\widehat{\chi})\,|_{\widehat{\tau}}$ is the $d \times d$ diagonal matrix with diagonal entries, $k = 1, \cdots, d$,

$$[\widehat{\Xi}_\varepsilon(\widehat{\chi})\,|_{\widehat{\tau}}]_{kk} := \begin{cases} \frac{\widehat{\chi}(\widehat{p}_k)-\widehat{\chi}(\widehat{p}_0)}{G'_\varepsilon(\widehat{\chi}(\widehat{p}_k))-G'_\varepsilon(\widehat{\chi}(\widehat{p}_0))} \equiv \frac{\chi(p_{j_k})-\chi(p_{j_0})}{G'_\varepsilon(\chi(p_{j_k}))-G'_\varepsilon(\chi(p_{j_0}))} & \\ \qquad = \frac{1}{G''_\varepsilon(\chi(\zeta))} \quad \text{for some } \zeta \text{ between } p_{j_k} \text{ and } p_{j_0} & \\ \qquad\qquad\qquad\qquad \text{if} \;\; \chi(p_{j_k}) \neq \chi(p_{j_0}), & \\ \frac{1}{G''_\varepsilon(\widehat{\chi}(\widehat{p}_0))} \equiv \frac{1}{G''_\varepsilon(\chi(p_{j_0}))} \qquad \text{if} \;\; \chi(p_{j_k}) = \chi(p_{j_0}). & \end{cases} \tag{2.4.30}$$

Under the assumptions (A), for any given $\varepsilon \in (0, e^{-1})$ we consider the following fully discrete finite element approximation of $(\mathrm{P}_{M,\varepsilon})$:

$(\mathbf{P}^{h,\Delta t}_{M,\varepsilon})$ For $n \geq 1$ find $\{U^n_\varepsilon, V^n_\varepsilon\} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$\left(\frac{U^n_\varepsilon - U^{n-1}_\varepsilon}{\Delta t_n}, \chi\right)^h + \left(D\,\nabla U^n_\varepsilon + \Lambda_\varepsilon(U^n_\varepsilon)\,\nabla\,(U^n_\varepsilon + V^n_\varepsilon)\,, \nabla\chi\right)$$
$$= (U^n_\varepsilon - U^n_\varepsilon\,\phi_\varepsilon(U^{n-1}_\varepsilon) - \phi_\varepsilon(U^n_\varepsilon)\,\psi_\varepsilon(V^{n-1}_\varepsilon), \chi)^h\,, \tag{2.4.31a}$$

$$\left(\frac{V^n_\varepsilon - V^{n-1}_\varepsilon}{\Delta t_n}, \chi\right)^h + \left(D\,\nabla V^n_\varepsilon + \Xi_\varepsilon(V^n_\varepsilon)\,\nabla\,(U^n_\varepsilon + V^n_\varepsilon)\,, \nabla\chi\right)$$
$$= \left(\gamma\,V^n_\varepsilon - \psi_\varepsilon(V^n_\varepsilon)\,\left[\phi_\varepsilon(U^{n-1}_\varepsilon) + \psi_\varepsilon(V^{n-1}_\varepsilon)\right], \chi\right)^h, \tag{2.4.31b}$$

where $U^0_\varepsilon \in S^h$ and $V^0_\varepsilon \in S^h$ are given approximations of u^0 and v^0 respectively.

Before we prove existence of the approximate solutions, in the following subsection, we provide some lemmata which will be important in the analysis of the approximation problem $(\mathrm{P}^{h,\Delta t}_{M,\varepsilon})$. The proofs of these lemmata will be based on arguments considered in [11] and [12].

Lemma 2.4.3 Let the assumptions (A) hold. Then for any given $\varepsilon \in (0, e^{-1})$ the functions $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ and $\Xi_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ satisfy, respectively, for $a.e.$ in Ω

$$\varepsilon\,\xi^T\xi \leq \xi^T\Lambda_\varepsilon(\chi)\xi \leq M\,\xi^T\xi \qquad \forall\;\xi \in \mathbb{R}^d,\;\forall\;\chi \in S^h, \tag{2.4.32}$$

$$\varepsilon\,\xi^T\xi \leq \xi^T\Xi_\varepsilon(\chi)\xi \leq \varepsilon^{-1}\,\xi^T\xi \qquad \forall\;\xi \in \mathbb{R}^d,\;\forall\;\chi \in S^h. \tag{2.4.33}$$

Proof: Let $\chi \in S^h$ and $\tau \in \mathcal{T}^h$. It follows, on noting the symmetry of $\Lambda_\varepsilon(\chi)\,|_\tau$ and $\widehat{\Lambda}_\varepsilon(\widehat{\chi})\,|_{\widehat{\tau}}$, $R_\tau^T \equiv R_\tau^{-1}$ and (2.4.24), that $\Lambda_\varepsilon(\chi)\,|_\tau$ and $\widehat{\Lambda}_\varepsilon(\widehat{\chi})\,|_{\widehat{\tau}}$ possess the same eigenvalues. In particular, we have

$$\|\Lambda_\varepsilon(\chi)\,|_\tau\,\| = \|\widehat{\Lambda}_\varepsilon(\widehat{\chi})\,|_{\widehat{\tau}}\,\| \qquad \forall\, \tau \in \mathcal{T}^h,$$

where $\|\,.\,\|$ denotes the spectral norm on $\mathbb{R}^{d\times d}$. Noting this, (2.4.25) and (2.3.5) yields the result (2.4.32), (see Theorem 9.12 in [18]). Similarly to (2.4.32), the result (2.4.33) follows from (2.4.29), (2.4.30) and (2.3.11). □

Lemma 2.4.4 Let the assumptions (A) hold. Then for any given $\varepsilon \in (0, e^{-1})$ the functions $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ and $\Xi_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ are continuous in the following sense. For all $\chi_1\,,\ \chi_2 \in S^h$ and $\tau \in \mathcal{T}^h$

$$\begin{aligned}
&\|(\Lambda_\varepsilon(\chi_1) - \Lambda_\varepsilon(\chi_2))\,|_\tau\,\| \\
&\qquad \leq \max_{s\in\mathbb{R}}\,[F_\varepsilon''(s)]\,\max_{s\in\mathbb{R}}\,[\phi_\varepsilon(s)]\,\max_{k=1,\cdots,d}\,[\,|\chi_1(p_{j_k}) - \chi_2(p_{j_k})| + |\chi_1(p_{j_0}) - \chi_2(p_{j_0})|\,] \\
&\qquad \leq \tfrac{2\,M}{\varepsilon}\,\|\chi_1 - \chi_2\|_{0,\infty}\,, && (2.4.34)\\
&\|(\Xi_\varepsilon(\chi_1) - \Xi_\varepsilon(\chi_2))\,|_\tau\,\| \\
&\qquad \leq \max_{s\in\mathbb{R}}\,[G_\varepsilon''(s)]\,\max_{s\in\mathbb{R}}\,[\psi_\varepsilon(s)]\,\max_{k=1,\cdots,d}\,[\,|\chi_1(p_{j_k}) - \chi_2(p_{j_k})| + |\chi_1(p_{j_0}) - \chi_2(p_{j_0})|\,] \\
&\qquad \leq \tfrac{2}{\varepsilon^2}\,\|\chi_1 - \chi_2\|_{0,\infty}\,. && (2.4.35)
\end{aligned}$$

Proof: We provide the proof of (2.4.34) which can be easily modified to show (2.4.35). On noting the construction of Λ_ε we have for any $\chi_1\,,\ \chi_2 \in S^h$ and $\tau \in \mathcal{T}^h$ that

$$\begin{aligned}
\|(\Lambda_\varepsilon(\chi_1) - \Lambda_\varepsilon(\chi_2))\,|_\tau\,\| &= \|(\widehat{\Lambda}_\varepsilon(\widehat{\chi}_1) - \widehat{\Lambda}_\varepsilon(\widehat{\chi}_2))\,|_{\widehat{\tau}}\,\| \\
&= \max_{k=1,\cdots,d}\,|[\widehat{\Lambda}_\varepsilon(\widehat{\chi}_1) - \widehat{\Lambda}_\varepsilon(\widehat{\chi}_2)]_{kk}\,|_{\widehat{\tau}}\,| := \max_{k=1,\cdots,d}\,\mathbf{I}_k\,, \qquad (2.4.36)
\end{aligned}$$

where we generally set for any $k = 1, \cdots, d$

$$\mathbf{I}_k = |\phi_\varepsilon(\xi_1) - \phi_\varepsilon(\xi_2)|$$

for some ξ_1 between (or equal to) $\chi_1(p_{j_0})$ and $\chi_1(p_{j_k})$ and for some ξ_2 between (or equal to) $\chi_2(p_{j_0})$ and $\chi_2(p_{j_k})$.

We have two cases:

(1) $\chi_1(p_{j_0}) = \chi_1(p_{j_k})$ or $\chi_2(p_{j_0}) = \chi_2(p_{j_k})$.

(2) $\chi_1(p_{j_0}) \neq \chi_1(p_{j_k})$ and $\chi_2(p_{j_0}) \neq \chi_2(p_{j_k})$ with

either $(2a)$ $\chi_1(p_{j_0}) = \chi_2(p_{j_k})$ and $\chi_2(p_{j_0}) = \chi_1(p_{j_k})$

or $(2b)$ $\chi_1(p_{j_0}) \neq \chi_2(p_{j_k})$ or $\chi_2(p_{j_0}) \neq \chi_1(p_{j_k})$.

The cases (1) and $(2a)$ can be easily treated on noting the Lipschitz continuity of ϕ_ε since in both cases we have

$$\begin{aligned}\mathbf{I}_k &= |\phi_\varepsilon(\xi_1) - \phi_\varepsilon(\xi_2)| \leq |\xi_1 - \xi_2| \\ &\leq \max\{|\chi_1(p_{j_0}) - \chi_2(p_{j_0})|, |\chi_1(p_{j_0}) - \chi_2(p_{j_k})|, \\ &\qquad\qquad |\chi_1(p_{j_k}) - \chi_2(p_{j_0})|, |\chi_1(p_{j_k}) - \chi_2(p_{j_k})|\} \\ &= \max\{|\chi_1(p_{j_0}) - \chi_2(p_{j_0})|, |\chi_1(p_{j_k}) - \chi_2(p_{j_k})|\} \, . \end{aligned} \tag{2.4.37}$$

We now consider the case $(2b)$ which requires some technical calculations. Without loss of generality, we assume that $\chi_1(p_{j_0}) \neq \chi_2(p_{j_k})$ and we set

$$\phi_\varepsilon(\xi_{1,2}) := \frac{\chi_1(p_{j_0}) - \chi_2(p_{j_k})}{F'_\varepsilon(\chi_1(p_{j_0})) - F'_\varepsilon(\chi_2(p_{j_k}))} \quad , \quad \xi_{1,2} \text{ between } \chi_1(p_{j_0}) \text{ and } \chi_2(p_{j_k}) \, .$$

We have

$$\begin{aligned}\mathbf{I}_k &= |\phi_\varepsilon(\xi_1) - \phi_\varepsilon(\xi_2)| \leq |\phi_\varepsilon(\xi_1) - \phi_\varepsilon(\xi_{1,2})| + |\phi_\varepsilon(\xi_{1,2}) - \phi_\varepsilon(\xi_2)| \\ &= \left| \tfrac{\chi_1(p_{j_0}) - \chi_1(p_{j_k})}{F'_\varepsilon(\chi_1(p_{j_0})) - F'_\varepsilon(\chi_1(p_{j_k}))} - \tfrac{\chi_1(p_{j_0}) - \chi_2(p_{j_k})}{F'_\varepsilon(\chi_1(p_{j_0})) - F'_\varepsilon(\chi_2(p_{j_k}))} \right| \\ &\qquad + \left| \tfrac{\chi_1(p_{j_0}) - \chi_2(p_{j_k})}{F'_\varepsilon(\chi_1(p_{j_0})) - F'_\varepsilon(\chi_2(p_{j_k}))} - \tfrac{\chi_2(p_{j_0}) - \chi_2(p_{j_k})}{F'_\varepsilon(\chi_2(p_{j_0})) - F'_\varepsilon(\chi_2(p_{j_k}))} \right| \\ &:= \mathbf{I}_{k,1} + \mathbf{I}_{k,2} \, . \end{aligned} \tag{2.4.38}$$

We deal with the terms $\mathbf{I}_{k,1}$ and $\mathbf{I}_{k,2}$ separately. If $\chi_1(p_{j_k}) = \chi_2(p_{j_k})$ then $\mathbf{I}_{k,1} = 0$. Otherwise, we set

$$\phi_\varepsilon(\xi_0) := \frac{\chi_1(p_{j_k}) - \chi_2(p_{j_k})}{F'_\varepsilon(\chi_1(p_{j_k})) - F'_\varepsilon(\chi_2(p_{j_k}))} \quad , \quad \xi_0 \text{ between } \chi_1(p_{j_k}) \text{ and } \chi_2(p_{j_k}) \, .$$

For any $\chi_1(p_{j_0}),\ \chi_1(p_{j_k}),\ \chi_2(p_{j_k}) \in \mathbb{R}$, there are three possibilities

$$\text{either}\quad |\chi_1(p_{j_k}) - \chi_2(p_{j_k})| = |\chi_1(p_{j_k}) - \chi_1(p_{j_0})| + |\chi_1(p_{j_0}) - \chi_2(p_{j_k})| \tag{2.4.39a}$$

$$\text{or}\quad |\chi_1(p_{j_k}) - \chi_2(p_{j_k})| = |\chi_1(p_{j_k}) - \chi_1(p_{j_0})| - |\chi_1(p_{j_0}) - \chi_2(p_{j_k})| \tag{2.4.39b}$$

$$\text{or}\quad |\chi_1(p_{j_k}) - \chi_2(p_{j_k})| = |\chi_2(p_{j_k}) - \chi_1(p_{j_0})| - |\chi_1(p_{j_0}) - \chi_1(p_{j_k})|\,. \tag{2.4.39c}$$

If (2.4.39a) holds then we obtain from the Lipschitz continuity of ϕ_ε that

$$\begin{aligned}
\mathbf{I}_{k,1} := |\phi_\varepsilon(\xi_1) - \phi_\varepsilon(\xi_{1,2})| &\le |\xi_1 - \xi_{1,2}| \\
&\le |\xi_1 - \chi_1(p_{j_0})| + |\chi_1(p_{j_0}) - \xi_{1,2}| \\
&\le |\chi_1(p_{j_k}) - \chi_1(p_{j_0})| + |\chi_1(p_{j_0}) - \chi_2(p_{j_k})| \\
&= |\chi_1(p_{j_k}) - \chi_2(p_{j_k})|\,.
\end{aligned} \tag{2.4.40}$$

Suppose that (2.4.39b) holds. We note that

$$\begin{aligned}
|\xi_{1,2} - \xi_0| &\le |\xi_{1,2} - \chi_2(p_{j_k})| + |\chi_2(p_{j_k}) - \xi_0| \\
&\le |\chi_1(p_{j_0}) - \chi_2(p_{j_k})| + |\chi_2(p_{j_k}) - \chi_1(p_{j_k})| \\
&= |\chi_1(p_{j_k}) - \chi_1(p_{j_0})|\,.
\end{aligned} \tag{2.4.41}$$

Hence, after some calculations, it follows from the Lipschitz continuity of ϕ_ε and (2.4.41) that

$$\begin{aligned}
\mathbf{I}_{k,1} :&= \left| \tfrac{\chi_1(p_{j_0}) - \chi_1(p_{j_k})}{F'_\varepsilon(\chi_1(p_{j_0})) - F'_\varepsilon(\chi_1(p_{j_k}))} - \tfrac{\chi_1(p_{j_0}) - \chi_2(p_{j_k})}{F'_\varepsilon(\chi_1(p_{j_0})) - F'_\varepsilon(\chi_2(p_{j_k}))} \right| \\
&= \left| \tfrac{F'_\varepsilon(\chi_1(p_{j_k})) - F'_\varepsilon(\chi_2(p_{j_k}))}{\chi_1(p_{j_k}) - \chi_2(p_{j_k})} \right| \times \left| \tfrac{\chi_1(p_{j_0}) - \chi_1(p_{j_k})}{F'_\varepsilon(\chi_1(p_{j_0})) - F'_\varepsilon(\chi_1(p_{j_k}))} \right| \times \left| \tfrac{\chi_1(p_{j_k}) - \chi_2(p_{j_k})}{\chi_1(p_{j_k}) - \chi_1(p_{j_0})} \right| \\
&\qquad \times \left| \tfrac{\chi_1(p_{j_k}) - \chi_2(p_{j_k})}{F'_\varepsilon(\chi_1(p_{j_k})) - F'_\varepsilon(\chi_2(p_{j_k}))} - \tfrac{\chi_1(p_{j_0}) - \chi_2(p_{j_k})}{F'_\varepsilon(\chi_1(p_{j_0})) - F'_\varepsilon(\chi_2(p_{j_k}))} \right| \\
&= F''_\varepsilon(\xi_0)\, \phi_\varepsilon(\xi_1)\ |\chi_1(p_{j_k}) - \chi_2(p_{j_k})| \left| \tfrac{\phi_\varepsilon(\xi_{1,2}) - \phi_\varepsilon(\xi_0)}{\chi_1(p_{j_k}) - \chi_1(p_{j_0})} \right| \\
&\le F''_\varepsilon(\xi_0)\, \phi_\varepsilon(\xi_1)\ |\chi_1(p_{j_k}) - \chi_2(p_{j_k})|\ .
\end{aligned} \tag{2.4.42}$$

Finally, if (2.4.39c) holds we obtain, similarly to (2.4.42), that

$$\begin{aligned}
\mathbf{I}_{k,1} &= F''_\varepsilon(\xi_0)\, \phi_\varepsilon(\xi_{1,2})\ |\chi_1(p_{j_k}) - \chi_2(p_{j_k})| \left| \tfrac{\phi_\varepsilon(\xi_1) - \phi_\varepsilon(\xi_0)}{\chi_2(p_{j_k}) - \chi_1(p_{j_0})} \right| \\
&\le F''_\varepsilon(\xi_0)\, \phi_\varepsilon(\xi_{1,2})\ |\chi_1(p_{j_k}) - \chi_2(p_{j_k})|\ .
\end{aligned} \tag{2.4.43}$$

Combining (2.4.40), (2.4.42) and (2.4.43) yields

$$\mathbf{I}_{k,1} \leq \max_{s\in\mathbb{R}} [F_\varepsilon''(s)] \, \max_{s\in\mathbb{R}} [\phi_\varepsilon(s)] \, |\chi_1(p_{j_k}) - \chi_2(p_{j_k})| \, . \tag{2.4.44}$$

Similarly to $\mathbf{I}_{k,1}$, we can show that

$$\mathbf{I}_{k,2} \leq \max_{s\in\mathbb{R}} [F_\varepsilon''(s)] \, \max_{s\in\mathbb{R}} [\phi_\varepsilon(s)] \, |\chi_1(p_{j_0}) - \chi_2(p_{j_0})| \, . \tag{2.4.45}$$

Thus, the result (2.4.34) follows by combining (2.4.36), (2.4.37), (2.4.38), (2.4.44), (2.4.45) and (2.3.5). □

Lemma 2.4.5 Let the assumptions (A) hold. Then for any given $\varepsilon \in (0, e^{-1})$ and for any $\chi \in S^h$ and $\tau \in \mathcal{T}^h$ the functions $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ and $\Xi_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ satisfy

$$\max_{x\in\tau} \|\Lambda_\varepsilon(\chi(x)) - \phi_\varepsilon(\chi(x))\,\mathcal{I}\| \leq h_\tau \, |\nabla\chi|_\tau \, | \, , \tag{2.4.46}$$

$$\max_{x\in\tau} \|\Xi_\varepsilon(\chi(x)) - \psi_\varepsilon(\chi(x))\,\mathcal{I}\| \leq h_\tau \, |\nabla\chi|_\tau \, | \, , \tag{2.4.47}$$

where $\mathcal{I}$ is the $d \times d$ identity matrix.

Proof: From (2.4.24), (2.4.25) and 1-Lipschitz continuity of ϕ_ε we obtain that

$$\begin{aligned}
\max_{x\in\tau} \|\Lambda_\varepsilon(\chi(x)) - \phi_\varepsilon(\chi(x))\,\mathcal{I}\| &= \max_{x\in\tau} \max_{k=1,\cdots,d} |\, [\widehat{\Lambda}_\varepsilon(\widehat{\chi})\,|_{\widehat{\tau}}]_{kk} - \phi_\varepsilon(\chi(x))\,| \\
&= \max_{x\in\tau} \max_{k=1,\cdots,d} |\, \phi_\varepsilon(\chi(\zeta_k)) - \phi_\varepsilon(\chi(x))\,| \qquad \zeta_k \in \overline{\tau} \\
&\leq \max_{x\in\tau} \max_{k=1,\cdots,d} |\, \chi(\zeta_k) - \chi(x)\,| \qquad \zeta_k \in \overline{\tau} \\
&\leq \max_{x\in\tau} \max_{k=1,\cdots,d} |\, \zeta_k - x\,|\,|\,\nabla\chi(x)\,| \qquad \zeta_k \in \overline{\tau} \\
&\leq h_\tau \, |\nabla\chi|_\tau \, | \, ,
\end{aligned}$$

which proves (2.4.46). Similarly to (2.4.46), the proof of (2.4.47) can be easily established on noting (2.4.29), (2.4.30) and the 1-Lipschitz continuity of the regularized function ψ_ε. □

In the following subsection we adapt the approach in Barrett and Nürnberg [12] and Barrett and Blowey [9] to prove the existence of the fully discrete approximations $\{U_\varepsilon^n, \; V_\varepsilon^n\}$ for $n = 1, \cdots, N$.

2.4.3 Existence of the approximations

In order to prove the existence of solution $\{U_\varepsilon^n,\ V_\varepsilon^n\}$, $n \geq 1$, of the system (2.4.31a)-(2.4.31b) for given $\{U_\varepsilon^{n-1},\ V_\varepsilon^{n-1}\}$, it is convenient to define the functions $A_u : S^h \times S^h \to S^h$ and $A_v : S^h \times S^h \to S^h$ such that for all $\chi \in S^h$

$$
\begin{aligned}
(A_u(U,V),\chi)^h &= \left(U - U_\varepsilon^{n-1},\chi\right)^h + \Delta t_n\ \left(D\,\nabla U + \Lambda_\varepsilon(U)\,\nabla\left(U+V\right),\nabla\chi\right) \\
&\quad - \Delta t_n \left(U - U\,\phi_\varepsilon(U_\varepsilon^{n-1}) - \phi_\varepsilon(U)\,\psi_\varepsilon(V_\varepsilon^{n-1}),\chi\right)^h, \qquad (2.4.48a)\\
(A_v(U,V),\chi)^h &= \left(V - V_\varepsilon^{n-1},\chi\right)^h + \Delta t_n\ \left(D\,\nabla V + \Xi_\varepsilon(V)\,\nabla\left(U+V\right),\nabla\chi\right) \\
&\quad - \Delta t_n \left(\gamma\,V - \psi_\varepsilon(V)\ \left[\phi_\varepsilon(U_\varepsilon^{n-1}) + \psi_\varepsilon(V_\varepsilon^{n-1})\right],\chi\right)^h, \qquad (2.4.48b)
\end{aligned}
$$

respectively. We first note that the continuous piecewise linear functions $A_u(U,V)$ and $A_v(U,V)$ can be defined uniquely in terms of their values at the nodal points $\mathcal{N}^h$. This can be seen by setting $\chi \equiv \varphi_j$, for $j = 0,\cdots,J$, in (2.4.48a,b) and then obtaining the following solvable square matrix systems

$$
\begin{aligned}
\widehat{M}\,A_u(U,V) &= S_1\,,\\
\widehat{M}\,A_v(U,V) &= S_2\,,
\end{aligned}
$$

where $\widehat{M}$ is the lumped mass matrix introduced in Subsection 4.2.1, and S_1 and S_2 are given vectors in terms of the nodal values of U, V, U_ε^{n-1} and V_ε^{n-1}. Thus, the functions A_u and A_v are well defined.

From (2.4.48a,b) we note that the problem $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ can be restated as:
For given $\{U_\varepsilon^0,\ V_\varepsilon^0\} \in S^h \times S^h$, find $\{U_\varepsilon^n,\ V_\varepsilon^n\} \in S^h \times S^h$, $n \geq 1$, such that

$$
A_u(U_\varepsilon^n, V_\varepsilon^n) = 0 \quad \text{and} \quad A_v(U_\varepsilon^n, V_\varepsilon^n) = 0\,. \qquad (2.4.49)
$$

Lemma 2.4.6 For any given $R > 0$, the functions $A_u : [S^h]_R^2 \to S^h$ and $A_v : [S^h]_R^2 \to S^h$ are continuous, where

$$
[S^h]_R^2 := \left\{\ \{\chi_1,\chi_2\} \in S^h \times S^h\ : |\chi_1|_h^2 + |\chi_2|_h^2 \leq R^2\ \right\}\,.
$$

Proof: Let $\{U_1, V_1\}\,,\ \{U_2, V_2\} \in [S^h]^2_R\,.$ It follows from (2.4.48a) that for all $\chi \in S^h$

$$
\begin{aligned}
(A_u(U_1,V_1) - A_u(U_2,V_2),\chi)^h &= (U_1 - U_2,\chi)^h \\
&\quad + \Delta t_n\ (D\,\nabla(U_1-U_2) + \Lambda_\varepsilon(U_1)\,\nabla\,(U_1+V_1) - \Lambda_\varepsilon(U_2)\,\nabla\,(U_2+V_2)\,,\nabla\chi) \\
&\quad - \Delta t_n\ \left((U_1-U_2)\,(1-\phi_\varepsilon(U_\varepsilon^{n-1})) - (\phi_\varepsilon(U_1)-\phi_\varepsilon(U_2))\,\psi_\varepsilon(V_\varepsilon^{n-1}),\chi\right)^h.
\end{aligned}
\tag{2.4.50}
$$

Choosing $\chi = A_u(U_1,V_1) - A_u(U_2,V_2)$ in (2.4.50) yields on noting the Cauchy-Schwarz inequality, (2.4.14) and (2.4.2) that

$$
\begin{aligned}
|A_u(U_1,V_1) - A_u(U_2,V_2)|_h &\leq C(h^{-1},\Delta t_n)\ \|\Lambda_\varepsilon(U_1)\,\nabla\,(U_1+V_1) - \Lambda_\varepsilon(U_2)\,\nabla\,(U_2+V_2)\|_0 \\
&\quad + \Delta t_n\ \left|(\phi_\varepsilon(U_1)-\phi_\varepsilon(U_2))\,\psi_\varepsilon(V_\varepsilon^{n-1})\right|_h + C(h^{-1}\,,\Delta t_n\,,M)\ |U_1-U_2|_h\,.
\end{aligned}
\tag{2.4.51}
$$

It follows from (2.3.11), (2.4.1) and the Lipschitz continuity of ϕ_ε that

$$
\left|(\phi_\varepsilon(U_1)-\phi_\varepsilon(U_2))\,\psi_\varepsilon(V_\varepsilon^{n-1})\right|_h \leq \tfrac{1}{\varepsilon}\ |\phi_\varepsilon(U_1)-\phi_\varepsilon(U_2)|_h \leq \tfrac{1}{\varepsilon}\ |U_1-U_2|_h\,. \tag{2.4.52}
$$

We also have from (2.4.14), (2.4.2), (2.4.34), (2.4.32) and (2.4.15) that

$$
\begin{aligned}
&\|\Lambda_\varepsilon(U_1)\,\nabla\,(U_1+V_1) - \Lambda_\varepsilon(U_2)\,\nabla\,(U_2+V_2)\|_0 \\
&\qquad \leq \|\Lambda_\varepsilon(U_1)-\Lambda_\varepsilon(U_2)\|_{0,\infty}\,|U_1|_1 + \|\Lambda_\varepsilon(U_2)\|_{0,\infty}\,|U_1-U_2|_1 \\
&\qquad\qquad + \|\Lambda_\varepsilon(U_1)-\Lambda_\varepsilon(U_2)\|_{0,\infty}\,|V_1|_1 + \|\Lambda_\varepsilon(U_2)\|_{0,\infty}\,|V_1-V_2|_1 \\
&\qquad \leq C(h^{-1})\ \|\Lambda_\varepsilon(U_1)-\Lambda_\varepsilon(U_2)\|_{0,\infty}\,(|U_1|_h + |V_1|_h) \\
&\qquad\qquad + C(h^{-1})\ \|\Lambda_\varepsilon(U_2)\|_{0,\infty}\,(|U_1-U_2|_h + |V_1-V_2|_h) \\
&\qquad \leq C(h^{-1},\,\varepsilon^{-1},\,M\,,R)\ \|U_1-U_2\|_{0,\infty} \\
&\qquad\qquad + C(h^{-1}\,,\,M)\ (|U_1-U_2|_h + |V_1-V_2|_h) \\
&\qquad \leq C(h^{-1},\,\varepsilon^{-1},\,M\,,R)\ (|U_1-U_2|_h + |V_1-V_2|_h)\,.
\end{aligned}
\tag{2.4.53}
$$

Combining (2.4.51), (2.4.52) and (2.4.53) yields that A_u is continuous. The continuity of A_v follows similarly to A_u on recalling (2.4.48b), (2.3.5), (2.3.11), (2.4.1), (2.4.35) and (2.4.33). $\square$

We now show the main result of this chapter where we establish the existence of a solution $\{U_\varepsilon^n,\ V_\varepsilon^n\}_{n=1}^N$ to $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$.

Theorem 2.4.7 Let the assumptions (A) hold, $D > 0$ and $\gamma > 1$. Let $\{U_\varepsilon^{n-1}, V_\varepsilon^{n-1}\} \in S^h \times S^h$ be a given solution to the $(n-1)$-th step of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ for some $n = 1, \cdots, N$. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ and for all $\Delta t_n > 0$ such that $\Delta t_n \leq \frac{1}{2\,\gamma+2}$, there exists a solution $\{U_\varepsilon^n, V_\varepsilon^n\} \in S^h \times S^h$ to the n-th step of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$.

Proof: At first, we recall that the proof is equivalent to the proof of existence of $\{U_\varepsilon^n, V_\varepsilon^n\} \in S^h \times S^h$ satisfies (2.4.49). An efficient approach to do that is by contradiction. Let R be a fixed positive number and assume that there does not exist $\{U, V\} \in [S^h]_R^2$ with $A_u(U,V) = A_v(U,V) = 0$. This assumption enables us to define a function $B : [S^h]_R^2 \to [S^h]_R^2$ such that

$$B(U,V) = \left(B_u(U,V), B_v(U,V)\right),$$

where $B_u(U,V)$ and $B_v(U,V)$ are given by

$$\begin{aligned} B_u(U,V) &:= \frac{-R\,A_u(U,V)}{|(A_u(U,V), A_v(U,V))|_{S^h\times S^h}}\,, \\ B_v(U,V) &:= \frac{-R\,A_v(U,V)}{|(A_u(U,V), A_v(U,V))|_{S^h\times S^h}}\,, \end{aligned} \tag{2.4.54}$$

where $|(\cdot,\cdot)|_{S^h\times S^h}$ is the standard norm on $S^h \times S^h$ defined by

$$|(\chi_1, \chi_2)|_{S^h\times S^h} := \left(|\chi_1|_h^2 + |\chi_2|_h^2\right)^{\frac{1}{2}}.$$

We note from the continuity of A_u and A_v , see Lemma 2.4.6 , that the function B is continuous. Hence, on recalling that $[S^h]_R^2$ is a convex and compact subset of $S^h \times S^h$, it follows from the Schauder's theorem (see Appendix A.1.1) that there exists $\{U, V\} \in [S^h]_R^2$ which is fixed point of B; that is

$$B(U,V) := \left(B_u(U,V), B_v(U,V)\right) = (U,V)\,.$$

We also note from (2.4.54) that the fixed point $\{U, V\}$ satisfies

$$|U|_h^2 + |V|_h^2 = |B_u(U,V)|_h^2 + |B_v(U,V)|_h^2 = R^2\,. \tag{2.4.55}$$

We now prove a contradiction for R sufficiently large. Choosing $\chi \equiv \pi^h[F'_\varepsilon(U)]$ in (2.4.48a) and $\chi \equiv \pi^h[G'_\varepsilon(V)]$ in (2.4.48b) yields on noting (2.4.3), (2.4.22b), (2.4.32), (2.4.27b) and (2.4.33) that

$$\begin{aligned}(A_u(U,V), F'_\varepsilon(U))^h &= \big(U - U_\varepsilon^{n-1}, F'_\varepsilon(U)\big)^h \\ &\quad + \Delta t_n \,\big(D\,[\Lambda_\varepsilon(U)]^{-1}\,\nabla U + \nabla(U+V), \nabla U\big) \\ &\quad - \Delta t_n \,(U - U\,\phi_\varepsilon(U_\varepsilon^{n-1}) - \phi_\varepsilon(U)\,\psi_\varepsilon(V_\varepsilon^{n-1}), F'_\varepsilon(U))^h \\ &\geq \big(U - U_\varepsilon^{n-1}, F'_\varepsilon(U)\big)^h + \big(\tfrac{D}{M} + 1\big)\,\Delta t_n\,|U|_1^2 + \Delta t_n\,(\nabla U, \nabla V) \\ &\quad - \Delta t_n \,(U - U\,\phi_\varepsilon(U_\varepsilon^{n-1}) - \phi_\varepsilon(U)\,\psi_\varepsilon(V_\varepsilon^{n-1}), F'_\varepsilon(U))^h \qquad (2.4.56a)\end{aligned}$$

and

$$\begin{aligned}(A_v(U,V), G'_\varepsilon(V))^h &= \big(V - V_\varepsilon^{n-1}, G'_\varepsilon(V)\big)^h \\ &\quad + \Delta t_n \,\big(D\,[\Xi_\varepsilon(V)]^{-1}\,\nabla V + \nabla(U+V), \nabla V\big) \\ &\quad - \Delta t_n \,(\gamma\,V - \psi_\varepsilon(V)\,\big[\phi_\varepsilon(U_\varepsilon^{n-1}) + \psi_\varepsilon(V_\varepsilon^{n-1})\big], G'_\varepsilon(V))^h \\ &\geq \big(V - V_\varepsilon^{n-1}, G'_\varepsilon(V)\big)^h + (D\,\varepsilon + 1)\,\Delta t_n\,|V|_1^2 + \Delta t_n\,(\nabla U, \nabla V) \\ &\quad - \Delta t_n \,(\gamma\,V - \psi_\varepsilon(V)\,\big[\phi_\varepsilon(U_\varepsilon^{n-1}) + \psi_\varepsilon(V_\varepsilon^{n-1})\big], G'_\varepsilon(V))^h.\end{aligned} \qquad (2.4.56b)$$

On noting Taylor's theorem for any $f \in C^2(\mathbb{R})$

$$(s_1 - s_2)\;f'(s_1) = f(s_1) - f(s_2) + \tfrac{(s_1-s_2)^2}{2}\;f''(\xi) \quad \text{for some } \xi \text{ between } s_1 \text{ and } s_2\,, \qquad (2.4.57)$$

we obtain from (2.3.5), (2.3.11) and (2.1.9) that

$$\begin{aligned}(U - U_\varepsilon^{n-1}, F'_\varepsilon(U))^h &\geq (F_\varepsilon(U) - F_\varepsilon(U_\varepsilon^{n-1}), 1)^h + \tfrac{1}{2\,M}\,|U - U_\varepsilon^{n-1}|_h^2 \\ &\geq (F_\varepsilon(U) - F_\varepsilon(U_\varepsilon^{n-1}), 1)^h + \tfrac{1}{4\,M}\,|U|_h^2 - \tfrac{1}{2\,M}\,|U_\varepsilon^{n-1}|_h^2\,, \qquad (2.4.58a) \\ (V - V_\varepsilon^{n-1}, G'_\varepsilon(V))^h &\geq (G_\varepsilon(V) - G_\varepsilon(V_\varepsilon^{n-1}), 1)^h + \tfrac{\varepsilon}{2}\,|V - V_\varepsilon^{n-1}|_h^2 \\ &\geq (G_\varepsilon(V) - G_\varepsilon(V_\varepsilon^{n-1}), 1)^h + \tfrac{\varepsilon}{4}\,|V|_h^2 - \tfrac{\varepsilon}{2}\,|V_\varepsilon^{n-1}|_h^2\,. \qquad (2.4.58b)\end{aligned}$$

It follows from (2.3.8), (2.3.9), (2.1.12), (2.1.10) and (2.3.6) that

$$
\begin{aligned}
&- \Delta t_n \left(U - U\,\phi_\varepsilon(U_\varepsilon^{n-1}) - \phi_\varepsilon(U)\,\psi_\varepsilon(V_\varepsilon^{n-1}), F_\varepsilon'(U)\right)^h \\
&\qquad = -\Delta t_n \,(U\,,\,F_\varepsilon'(U))^h + \Delta t_n \left(\phi_\varepsilon(U_\varepsilon^{n-1}), U\,F_\varepsilon'(U)\right)^h \\
&\qquad\qquad + \Delta t_n \left(\psi_\varepsilon(V_\varepsilon^{n-1}), \phi_\varepsilon(U)\,F_\varepsilon'(U)\right)^h \\
&\qquad \geq -\Delta t_n \,(2\,F_\varepsilon(U)+1,1)^h + \Delta t_n \left(\phi_\varepsilon(U_\varepsilon^{n-1}) + \psi_\varepsilon(V_\varepsilon^{n-1}), [U]_-\right)^h \\
&\qquad\qquad - \Delta t_n \left(\phi_\varepsilon(U_\varepsilon^{n-1}) + \psi_\varepsilon(V_\varepsilon^{n-1}), 1\right)^h \\
&\qquad \geq -2\,\Delta t_n\,(F_\varepsilon(U),1)^h - \tfrac{\Delta t_n}{\varepsilon}\,|[U]_-|_h^2 - \tfrac{\varepsilon\,\Delta t_n}{2}\left(|\phi_\varepsilon(U_\varepsilon^{n-1})|_h^2 + |\psi_\varepsilon(V_\varepsilon^{n-1})|_h^2\right) \\
&\qquad\qquad - \Delta t_n \left(\phi_\varepsilon(U_\varepsilon^{n-1}) + \psi_\varepsilon(V_\varepsilon^{n-1}) + 1, 1\right)^h \\
&\qquad \geq -4\,\Delta t_n\,(F_\varepsilon(U),1)^h - C(U_\varepsilon^{n-1}, V_\varepsilon^{n-1}).
\end{aligned}
\tag{2.4.59a}
$$

Similarly to (2.4.59a), we have from (2.3.14), (2.3.15), (2.1.12), (2.1.10) and (2.3.12) that

$$
\begin{aligned}
&- \Delta t_n \left(\gamma\,V - \psi_\varepsilon(V)\,\left[\phi_\varepsilon(U_\varepsilon^{n-1}) + \psi_\varepsilon(V_\varepsilon^{n-1})\right], G_\varepsilon'(V)\right)^h \\
&\qquad\qquad \geq -(2\,\gamma+2)\,\Delta t_n\,(G_\varepsilon(V),1)^h - C(U_\varepsilon^{n-1}, V_\varepsilon^{n-1}).
\end{aligned}
\tag{2.4.59b}
$$

Adding (2.4.56a,b) and noting (2.4.58a,b), (2.4.59a,b), the stated assumption on Δt_n and (2.4.55) yields for sufficiently large R that

$$
\begin{aligned}
&(A_u(U,V), F_\varepsilon'(U))^h + (A_v(U,V), G_\varepsilon'(V))^h \\
&\qquad \geq (1 - 4\,\Delta t_n)\,(F_\varepsilon(U),1)^h + (1 - (2\,\gamma+2)\,\Delta t_n)\,(G_\varepsilon(V),1)^h \\
&\qquad + \Delta t_n \left(|U|_1^2 + 2\,(\nabla U, \nabla V) + |V|_1^2\right) + \tfrac{1}{4\,M}\,|U|_h^2 + \tfrac{\varepsilon}{4}\,|V|_h^2 - C(U_\varepsilon^{n-1}, V_\varepsilon^{n-1}) \\
&\qquad \geq \Delta t_n\,|U+V|_1^2 + \left(|U|_h^2 + |V|_h^2\right)\,\min\{\tfrac{\varepsilon}{4}, \tfrac{1}{4\,M}\} - C(U_\varepsilon^{n-1}, V_\varepsilon^{n-1}) \\
&\qquad \geq R^2 \min\{\tfrac{\varepsilon}{4}, \tfrac{1}{4\,M}\} - C(U_\varepsilon^{n-1}, V_\varepsilon^{n-1}) > 0\,.
\end{aligned}
\tag{2.4.60}
$$

Noting that $\{U, V\}$ is fixed point of the function B, (2.4.54) and (2.4.60) yields for R sufficiently large that

$$
\begin{aligned}
(U, F_\varepsilon'(U))^h + (V, G_\varepsilon'(V))^h &= (B_u(U,V), F_\varepsilon'(U))^h + (B_v(U,V), G_\varepsilon'(V))^h \\
&= \frac{-R\,\left[(A_u(U,V), F_\varepsilon'(U))^h + (A_v(U,V), G_\varepsilon'(V))^h\right]}{|(A_u(U,V), A_v(U,V))|_{S^h\times S^h}} < 0\,.
\end{aligned}
\tag{2.4.61}
$$

Once again, it follows from (2.4.57), (2.3.5), and (2.3.11) that

$$(U, F'_\varepsilon(U))^h \geq (F_\varepsilon(U) - F_\varepsilon(0), 1)^h + \tfrac{1}{2\,M}\,|U|_h^2, \tag{2.4.62a}$$

$$(V, G'_\varepsilon(V))^h \geq (G_\varepsilon(V) - G_\varepsilon(0), 1)^h + \tfrac{\varepsilon}{2}\,|V|_h^2. \tag{2.4.62b}$$

Thus, combining (2.4.62a,b) and (2.4.55) yields on noting the non-negativity of $F_\varepsilon(s)$ and $G_\varepsilon(s)$ for R sufficiently large that

$$(U, F'_\varepsilon(U))^h + (V, G'_\varepsilon(V))^h \geq R^2 \min\{\tfrac{\varepsilon}{2}, \tfrac{1}{2\,M}\} - (2-\varepsilon)\,|\Omega| > 0\,, \tag{2.4.63}$$

which contradicts (2.4.61). This contradiction ensures that there exists $\{U_\varepsilon^n, V_\varepsilon^n\} \in S^h \times S^h$ satisfying $A_u(U_\varepsilon^n, V_\varepsilon^n) = A_v(U_\varepsilon^n, V_\varepsilon^n) = 0$. Equivalently, we have existence of a solution, which is $\{U_\varepsilon^n, V_\varepsilon^n\}$, to the n-th step of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$. $\square$

Chapter 3

The population model: Convergence and existence of a weak solution

In this chapter we prove the existence of a global weak solution to the system (2.2.4a)-(2.2.4d) by analysing the convergence of the fully discrete approximation problem ($\mathrm{P}_{M,\varepsilon}^{h,\Delta t}$). In addition to the tools presented in the last chapter, in Section 3.1 we introduce some notation which is required for the analysis of this chapter. In Section 3.2 we derive some stability bounds on the solutions of ($\mathrm{P}_{M,\varepsilon}^{h,\Delta t}$). Finally, in Section 3.3 we discuss the convergence of the approximate problem ($\mathrm{P}_{M,\varepsilon}^{h,\Delta t}$) and hence, we obtain the existence of a global weak solution to the system (2.2.4a)-(2.2.4d).

3.1 Auxiliary results

For later use in this chapter, we define P^h to be the discrete L^2-projection operator onto the finite dimensional space S^h where $P^h : L^2(\Omega) \to S^h$ is given by

$$(P^h\eta, \chi)^h = (\eta, \chi) \qquad \forall\, \chi \in S^h. \tag{3.1.1}$$

It follows from the Lax-Milgram theorem that $P^h\eta$ is a unique solution of (3.1.1). Furthermore, we note from (3.1.1) and (2.4.1) for any $\eta \in L^2(\Omega)$ that

$$(P^h\eta)(p_j) = \frac{(\eta,\varphi_j)}{\widehat{M}_{jj}} \qquad j = 0\,, \cdots, J\,.$$

Therefore, we have

$$\|P^h\eta\|_{0,\infty} \leq \|\eta\|_{0,\infty}\,. \qquad \forall\, \eta \in L^\infty(\Omega). \tag{3.1.2}$$

Recalling that we have a quasi-uniform family of partitioning $\mathcal{T}^h$, it holds for $m \in \{0,1\}$ that (see, e.g., [12]):

$$|(I-P^h)\eta|_{m,s} \leq C\,h^{1-m}\,|\eta|_{1,s} \quad \forall\, \eta \in W^{1,s}(\Omega) \quad \text{for any } s \in [2,\infty]. \tag{3.1.3}$$

For later purposes, we introduce for any $q \in (1,2]$ the "inverse Laplacian" operator $\mathcal{G}_q : \big(W^{1,q'}(\Omega)\big)' \to W^{1,q}(\Omega)$ such that

$$(\nabla\mathcal{G}_q v, \nabla\eta) + (\mathcal{G}_q v, \eta) = \langle v, \eta\rangle_{q'} \qquad \forall\, \eta \in W^{1,q'}(\Omega), \tag{3.1.4}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\langle\cdot,\cdot\rangle_{q'}$ denotes the duality pairing between $\big(W^{1,q'}(\Omega)\big)'$ and $W^{1,q'}(\Omega)$ that satisfies (see Appendix A.1.4):

$$\langle v, \eta\rangle_{q'} = (v,\eta) \qquad \forall\, v \in L^2(\Omega),\ \eta \in W^{1,q'}(\Omega). \tag{3.1.5}$$

The well-posedness of the operator $\mathcal{G}_q$ follows from the generalized Lax-Milgram theorem, see Appendix A.1.3, which additionally asserts the existence of a positive constant C such that

$$\|\mathcal{G}_q v\|_{1,q} \leq C\,\|v\|_{(W^{1,q'}(\Omega))'} \qquad \forall\, v \in (W^{1,q'}(\Omega))'. \tag{3.1.6}$$

For consistency of notation, when $q = 2$ the indices "q" and "q'" will be dropped on the above operator and duality pairing; that is $\mathcal{G} : (H^1(\Omega))' \to H^1(\Omega)$ defined by

$$(\nabla\mathcal{G}v, \nabla\eta) + (\mathcal{G}v, \eta) = \langle v, \eta\rangle \qquad \forall\, \eta \in H^1(\Omega), \tag{3.1.7}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ such that

$$\langle v, \eta\rangle = (v,\eta) \qquad \forall\, v \in L^2(\Omega),\ \eta \in H^1(\Omega). \tag{3.1.8}$$

Also, we note from (3.1.6) that

$$\|\mathcal{G}v\|_{H^1(\Omega)} \leq C\,\|v\|_{(H^1(\Omega))'} \qquad \forall\, v \in (H^1(\Omega))'. \tag{3.1.9}$$

We finally recall the following lemma, about the operator $\mathcal{G}_q$ for $q \in (1,2]$, which is a consequence of the quasi-uniform partitioning of $\mathcal{T}^h$:

Lemma 3.1.1 For any $q \in (1,2]$, it holds that

$$\|\chi\|_{0,q} \leq C\,h^{-1}\,\|\mathcal{G}_q\chi\|_{1,q} \quad \forall\, \chi \in S^h. \tag{3.1.10}$$

Proof: It follows from (3.1.5), (3.1.4), the Hölder's inequality, the Young's inequality and (2.4.13) for any $\chi \in S^h$ and for any $\alpha > 0$ that

$$\begin{aligned}
\|\chi\|_0^2 = \langle \chi, \chi\rangle_{q'} &= (\nabla \mathcal{G}_q\chi, \nabla\chi) + (\mathcal{G}_q\chi, \chi) \\
&\leq 2\,\|\mathcal{G}_q\chi\|_{1,q}\|\chi\|_{1,q'} \\
&\leq \alpha\,\|\mathcal{G}_q\chi\|_{1,q}^2 + \tfrac{C}{\alpha}\,h^{-2\,(1+d\,(\frac{1}{2}-\frac{1}{q'}))}\|\chi\|_0^2\,.
\end{aligned} \tag{3.1.11}$$

Choosing $\alpha = 2\,C\,h^{-2\,(1+d\,(\frac{1}{2}-\frac{1}{q'}))}$ in (3.1.11) yields, on again noting (2.4.13), that

$$\|\chi\|_{0,q} \leq C\,h^{d\,(\frac{1}{q}-\frac{1}{2})}\|\chi\|_0 \leq C\,h^{d\,(\frac{1}{q}-\frac{1}{2})-(1+d\,(\frac{1}{2}-\frac{1}{q'}))}\|\mathcal{G}_q\chi\|_{1,q} \leq C\,h^{-1}\,\|\mathcal{G}_q\chi\|_{1,q}\,.$$

$\square$

3.2 Stability estimates

In this section we obtain a discrete analogue of the *a priori* estimates in Lemma 2.3.1. We also prove some uniform bounds on the solution $\{U_\varepsilon^n, V_\varepsilon^n\}$, independent of the parameters ε, h and Δt_n, which are necessary to prove the convergence of the approximate problem.

The following estimate is discrete analogue of (2.3.24), and plays a key role to obtain important stability bounds of various norms of the approximate solutions.

Lemma 3.2.1 Let the assumptions (A) hold, $D > 0$ and $\gamma > 1$. Let $\{U_\varepsilon^{n-1},\, V_\varepsilon^{n-1}\} \in S^h \times S^h$ be given for some $n = 1, \cdots, N$. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ and for all $\Delta t_n > 0$ such that $\Delta t_n \leq \frac{1}{2\gamma+2}$, there exists a solution $\{U^n_\varepsilon, V^n_\varepsilon\} \in S^h \times S^h$ to the n-th step of $(\mathrm{P}^{h,\Delta t}_{M,\varepsilon})$ such that

$$\begin{aligned}(1-(2\,\gamma+2)\,\Delta t_n)\,(F_\varepsilon(U^n_\varepsilon)+G_\varepsilon(V^n_\varepsilon),1)^h+\tfrac{D}{M}\,\Delta t_n\,|U^n_\varepsilon|_1^2+\Delta t_n\,|U^n_\varepsilon+V^n_\varepsilon|_1^2 \\ \leq (1+6\,\Delta t_n)\,(F_\varepsilon(U^{n-1}_\varepsilon)+G_\varepsilon(V^{n-1}_\varepsilon),1)^h+C\,\Delta t_n\,. \qquad (3.2.1)\end{aligned}$$

Proof: The existence was demonstrated in Theorem 2.4.7. We now show that the solution $\{U^n_\varepsilon, V^n_\varepsilon\}$ satisfies (3.2.1). Choosing $\chi \equiv \Delta t_n\,\pi^h[F'_\varepsilon(U^n_\varepsilon)]$ as a test function in (2.4.31a) and $\chi \equiv \Delta t_n\,\pi^h[G'_\varepsilon(V^n_\varepsilon)]$ as a test function in (2.4.31b) yields, on noting (2.4.22b), (2.4.27b) and (2.4.3), the discrete analogue of (2.3.19)

$$\begin{aligned}&\left(U^n_\varepsilon-U^{n-1}_\varepsilon, F'_\varepsilon(U^n_\varepsilon)\right)^h+\Delta t_n\,\left(D\,[\Lambda_\varepsilon(U^n_\varepsilon)]^{-1}\,\nabla U^n_\varepsilon+\nabla(U^n_\varepsilon+V^n_\varepsilon),\nabla U^n_\varepsilon\right) \\ &\qquad =\Delta t_n\,(U^n_\varepsilon-U^n_\varepsilon\,\phi_\varepsilon(U^{n-1}_\varepsilon)-\phi_\varepsilon(U^n_\varepsilon)\,\psi_\varepsilon(V^{n-1}_\varepsilon),F'_\varepsilon(U^n_\varepsilon))^h, \qquad (3.2.2\text{a}) \\ &\left(V^n_\varepsilon-V^{n-1}_\varepsilon, G'_\varepsilon(V^n_\varepsilon)\right)^h+\Delta t_n\,\left(D\,[\Xi_\varepsilon(V^n_\varepsilon)]^{-1}\,\nabla V^n_\varepsilon+\nabla(U^n_\varepsilon+V^n_\varepsilon),\nabla V^n_\varepsilon\right) \\ &\qquad =\Delta t_n\,(\gamma\,V^n_\varepsilon-\psi_\varepsilon(V^n_\varepsilon)\,\left[\phi_\varepsilon(U^{n-1}_\varepsilon)+\psi_\varepsilon(V^{n-1}_\varepsilon)\right],G'_\varepsilon(V^n_\varepsilon))^h. \qquad (3.2.2\text{b})\end{aligned}$$

Similarly to (2.3.22), it follows from (2.3.5), (2.3.11), (2.3.8), (2.3.9), (2.1.11), (2.3.14), the Young's inequality and (2.3.6) that

$$\begin{aligned}&\Delta t_n\,(U^n_\varepsilon-U^n_\varepsilon\,\phi_\varepsilon(U^{n-1}_\varepsilon)-\phi_\varepsilon(U^n_\varepsilon)\,\psi_\varepsilon(V^{n-1}_\varepsilon),F'_\varepsilon(U^n_\varepsilon))^h \\ &\qquad \leq \Delta t_n\,(2\,F_\varepsilon(U^n_\varepsilon)+1,1)^h+\Delta t_n\,\left(\phi_\varepsilon(U^{n-1}_\varepsilon)+\psi_\varepsilon(V^{n-1}_\varepsilon),1\right)^h \\ &\qquad\qquad -\Delta t_n\,\left(\phi_\varepsilon(U^{n-1}_\varepsilon)+\psi_\varepsilon(V^{n-1}_\varepsilon),[U^n_\varepsilon]_-\right)^h \\ &\qquad \leq 2\,\Delta t_n\,(F_\varepsilon(U^n_\varepsilon),1)^h+2\,\Delta t_n\,(G_\varepsilon(V^{n-1}_\varepsilon),1)^h+\tfrac{\Delta t_n}{\varepsilon}\,|[U^n_\varepsilon]_-|_h^2 \\ &\qquad\qquad +\tfrac{\varepsilon\,\Delta t_n}{2}\,\left(|\phi_\varepsilon(U^{n-1}_\varepsilon)|_h^2+|\psi_\varepsilon(V^{n-1}_\varepsilon)|_h^2\right)+C(M,\,|\Omega|)\,\Delta t_n \\ &\qquad \leq 4\,\Delta t_n\,(F_\varepsilon(U^n_\varepsilon),1)^h+3\,\Delta t_n\,(G_\varepsilon(V^{n-1}_\varepsilon),1)^h+C(M,\,|\Omega|)\,\Delta t_n\,. \qquad (3.2.3\text{a})\end{aligned}$$

We also obtain, similarly to (2.3.23), from (2.3.5), (2.3.11), (2.3.14), (2.3.15), (2.1.11), the Young's inequality and (2.3.12) that

$$\begin{aligned}&\Delta t_n\,(\gamma\,V^n_\varepsilon-\psi_\varepsilon(V^n_\varepsilon)\,\left[\phi_\varepsilon(U^{n-1}_\varepsilon)+\psi_\varepsilon(V^{n-1}_\varepsilon)\right],G'_\varepsilon(V^n_\varepsilon))^h \\ &\qquad \leq (2\,\gamma+2)\,\Delta t_n\,(G_\varepsilon(V^n_\varepsilon),1)^h+3\,\Delta t_n\,(G_\varepsilon(V^{n-1}_\varepsilon),1)^h+C(M,\,|\Omega|,\,\gamma)\,\Delta t_n\,. \\ &\qquad (3.2.3\text{b})\end{aligned}$$

Combining (3.2.2a), (3.2.3a) and the first inequality in (2.4.58a) leads to

$$(1 - 4\,\Delta t_n)\,(F_\varepsilon(U_\varepsilon^n), 1)^h + \Delta t_n\,\left(D\,[\Lambda_\varepsilon(U_\varepsilon^n)]^{-1}\,\nabla U_\varepsilon^n + \nabla(U_\varepsilon^n + V_\varepsilon^n), \nabla U_\varepsilon^n\right)$$
$$\leq (F_\varepsilon(U_\varepsilon^{n-1}), 1)^h + 3\,\Delta t_n\,(G_\varepsilon(V_\varepsilon^{n-1}), 1)^h + C\,\Delta t_n\,. \quad (3.2.4a)$$

Combining (3.2.2b), (3.2.3b) and the first inequality in (2.4.58b) gives

$$(1 - (2\,\gamma + 2)\,\Delta t_n)\,(G_\varepsilon(V_\varepsilon^n), 1)^h + \Delta t_n\,\left(D\,[\Xi_\varepsilon(V_\varepsilon^n)]^{-1}\,\nabla V_\varepsilon^n + \nabla(U_\varepsilon^n + V_\varepsilon^n), \nabla V_\varepsilon^n\right)$$
$$\leq (1 + 3\,\Delta t_n)\,(G_\varepsilon(V_\varepsilon^{n-1}), 1)^h + C\,\Delta t_n\,. \quad (3.2.4b)$$

Hence, the estimate (3.2.1) follows by summing (3.2.4a) and (3.2.4b) on noting (2.4.32), (2.4.33), $F_\varepsilon(s) \geq 0$, $G_\varepsilon(s) \geq 0$ and that $\gamma > 1$. □

Lemma 3.2.2 Let the assumptions of Lemma 3.2.1 hold and let u^0, $v^0 \in L^\infty(\Omega)$ with $u^0(x)$, $v^0(x) \geq 0$ for a.e. $x \in \Omega$. Let either $U_\varepsilon^0 \equiv P^h u^0$ and $V_\varepsilon^0 \equiv P^h v^0$; or $U_\varepsilon^0 \equiv \pi^h u^0$ and $V_\varepsilon^0 \equiv \pi^h v^0$ if u^0, $v^0 \in C(\overline{\Omega})$ [1]. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ and for all $\Delta t > 0$ such that $\Delta t \leq \frac{1-\delta}{2\,\gamma+2}$, for some $\delta \in (0,1)$, the problem $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ possesses a solution $\{U_\varepsilon^n, V_\varepsilon^n\}_{n=1}^N$ satisfying

$$\max_{n=1,\cdots,N}\left[(F_\varepsilon(U_\varepsilon^n) + G_\varepsilon(V_\varepsilon^n), 1)^h + \varepsilon^{-1}\|\pi^h[U_\varepsilon^n]_-\|_0^2 + \varepsilon^{-1}\|\pi^h[V_\varepsilon^n]_-\|_0^2 + \|U_\varepsilon^n\|_0^2 + \|V_\varepsilon^n\|_{0,1}\right]$$
$$+ \sum_{n=1}^N \Delta t_n\,\|U_\varepsilon^n + V_\varepsilon^n\|_1^2 + \sum_{n=1}^N \Delta t_n\,\|U_\varepsilon^n\|_1^2 + \sum_{n=1}^N \Delta t_n\,\|V_\varepsilon^n\|_1^2 \leq C\,. \quad (3.2.5)$$

Proof: It follows immediately from (3.2.1) and our assumptions on Δt, for $n = 1, \cdots, N$, that

$$(F_\varepsilon(U_\varepsilon^n) + G_\varepsilon(V_\varepsilon^n), 1)^h \leq \left(1 + \tfrac{2\,(\gamma+4)\,\Delta t_n}{\delta}\right)(F_\varepsilon(U_\varepsilon^{n-1}) + G_\varepsilon(V_\varepsilon^{n-1}), 1)^h + \tfrac{C}{\delta}\,\Delta t_n$$
$$\leq e^{\frac{2\,(\gamma+4)\,\Delta t_n}{\delta}}(F_\varepsilon(U_\varepsilon^{n-1}) + G_\varepsilon(V_\varepsilon^{n-1}), 1)^h + \tfrac{C}{\delta}\,\Delta t_n\,. \quad (3.2.6)$$

On noting the assumptions on the initial data $\{u^0, v^0\}$, (2.3.4a), (2.3.10a), the definition of π^h and (3.1.2), we have that

$$(F_\varepsilon(U_\varepsilon^0), 1)^h + (G_\varepsilon(V_\varepsilon^0), 1)^h \leq C\,. \quad (3.2.7)$$

[1] On recalling the definitions of P^h and π^h we have, for non-negative initial data u^0 and v^0, that U_ε^0, $V_\varepsilon^0 \geq 0$.

Combining (3.2.6), (3.2.7) yields that

$$\max_{n=1,\cdots,N}\left[(F_\varepsilon(U_\varepsilon^n)+G_\varepsilon(V_\varepsilon^n),1)^h\right]\le C\,e^{\frac{2\,(\gamma+4)\,T}{\delta}}\,\left[T+(F_\varepsilon(U_\varepsilon^0)+G_\varepsilon(V_\varepsilon^0),1)^h\right]$$
$$\le C\,. \tag{3.2.8}$$

It follows from (2.4.2), (2.3.6), (2.3.7) and (3.2.8) for $n=1,\cdots,N$ that

$$\|U_\varepsilon^n\|_0^2\le|U_\varepsilon^n|_h^2=\left((U_\varepsilon^n)^2,1\right)^h\le 4M\,(F_\varepsilon(U_\varepsilon^n),1)^h+6M^2|\Omega|\le C. \tag{3.2.9}$$

Choosing $\chi\equiv 1$ in (2.4.31b) and noting the positivity of $\phi_\varepsilon(s)$ and $\psi_\varepsilon(s)$ yields, under the considered assumptions on the parameter Δt, that for $n=1,\cdots,N$

$$(V_\varepsilon^n,1)^h\le\tfrac{1}{1-\gamma\,\Delta t_n}\,(V_\varepsilon^{n-1},1)^h\le\left(1+\tfrac{\gamma\,\Delta t_n}{\delta}\right)(V_\varepsilon^{n-1},1)^h\le e^{\frac{\gamma\,\Delta t_n}{\delta}}\,(V_\varepsilon^{n-1},1)^h\,. \tag{3.2.10}$$

Hence, if follows from (3.2.10), the definition of the interpolation π^h, (3.1.2) and the assumptions on v^0 that

$$\max_{n=1,\cdots,N}(V_\varepsilon^n,1)^h\le e^{\frac{\gamma T}{\delta}}\left(V_\varepsilon^0,1\right)^h\le|\Omega|\,e^{\frac{\gamma T}{\delta}}\,\|v^0\|_{0,\infty}\le C\,. \tag{3.2.11}$$

Observing that $|s|=s-2\,[s]_-$, (3.2.11) and the Young's inequality yields for $n=1,\cdots,N$ that

$$\begin{aligned}\|V_\varepsilon^n\|_{0,1}=(|V_\varepsilon^n|,1)&\le\left(\pi^h|V_\varepsilon^n|,1\right)\\&\le\left(V_\varepsilon^n-2\,\pi^h\,[V_\varepsilon^n]_-\,,1\right)\\&=(V_\varepsilon^n,1)^h-2\,\left(\pi^h\,[V_\varepsilon^n]_-\,,1\right)\\&\le C\,\left(1+\|\pi^h[V_\varepsilon^n]_-\|_0^2\right)\,.\end{aligned} \tag{3.2.12}$$

From (2.4.2), (2.4.3), (2.3.6), (2.3.12) and (3.2.8) we obtain, after recalling that $s=[s]_++[s]_-$ and $F_\varepsilon(s)\,,\,G_\varepsilon(s)\ge 0$, that for $n=1,\cdots,N$

$$\begin{aligned}\|\pi^h[U_\varepsilon^n]_-\|_0^2\le|\pi^h[U_\varepsilon^n]_-|_h^2&=\left([U_\varepsilon^n]_-^2,1\right)^h\\&\le 2\,\varepsilon\,(F_\varepsilon(U_\varepsilon^n),1)^h\le C\,\varepsilon\,,\end{aligned} \tag{3.2.13a}$$

$$\begin{aligned}\|\pi^h[V_\varepsilon^n]_-\|_0^2\le|\pi^h[V_\varepsilon^n]_-|_h^2&=\left([V_\varepsilon^n]_-^2,1\right)^h\\&\le 2\,\varepsilon\,(G_\varepsilon(V_\varepsilon^n),1)^h\le C\,\varepsilon\,.\end{aligned} \tag{3.2.13b}$$

We now note that the bounds $1 \to 5$ in (3.2.5) follow by combining (3.2.8), (3.2.9), (3.2.12) and (3.2.13a,b). The sixth and the seventh bounds in (3.2.5) follow by summing (3.2.1) over n, with the aid of (3.2.7), (3.2.8), the Poincaré inequality and the bounds $(4-5)$ in (3.2.5). Finally, the last bound in (3.2.5) follows immediately from the triangle inequality on noting the sixth and the seventh bounds in (3.2.5). $\square$

Remark 3.2.1 We mention that the first, the sixth and the seventh bounds in (3.2.5) are discrete analogues of the estimates in (2.3.17). The second and the third bounds in (3.2.5) are discrete analogues of the estimates in (2.3.18).

Theorem 3.2.3 Let the assumptions of Lemma 3.2.2 hold. In addition, let $\{\Delta t_n\}_{n=1}^{N}$ be such that

$$\Delta t_n \leq C\, \Delta t_{n-1} \qquad \forall\, n = 2, \cdots, N.$$

Then a solution $\{U_\varepsilon^n, V_\varepsilon^n\}_{n=1}^N$ to $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ satisfies

$$\begin{aligned}\sum_{n=1}^{N} \Delta t_n & \left[\|U_\varepsilon^n\|_{0,\alpha}^{\alpha} + \|V_\varepsilon^n\|_{0,\beta}^{\beta} + \|\psi_\varepsilon(V_\varepsilon^n)\|_{0,\beta}^{\beta} + \|\pi^h\psi_\varepsilon(V_\varepsilon^n)\|_{0,\beta}^{\beta} + \|\Xi_\varepsilon(V_\varepsilon^n)\|_{0,\beta}^{\beta}\right] \\ & + \sum_{n=1}^{N} \Delta t_n \left[\|\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t_n}\|_{(H^1(\Omega))'}^2 + \|\tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\Delta t_n}\|_{(W^{1,q'}(\Omega))'}^q \right] \\ & \quad + \sum_{n=1}^{N} \Delta t_n \left[\|\mathcal{G}[\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t_n}]\|_1^2 + \|\mathcal{G}_q[\tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\Delta t_n}]\|_{1,q}^q \right] \leq C\,, \end{aligned} \tag{3.2.14}$$

where $\alpha = \frac{2\,(d+2)}{d}$, $\beta = \frac{2\,(d+1)}{d}$, $q = \frac{2\,(d+1)}{2\,d+1}$ and $q' = 2\,(d+1)$.

Proof: It follows from the Sobolev interpolation theorem (2.1.1) and the fourth bound in (3.2.5) for $n = 1, \cdots, N$ that

$$\|U_\varepsilon^n\|_{0,\alpha}^{\alpha} \leq C\, \|U_\varepsilon^n\|_0^{\alpha-2} \|U_\varepsilon^n\|_1^2 \leq C\, \|U_\varepsilon^n\|_1^2, \tag{3.2.15}$$

where $\alpha\, d\, (\frac{1}{2} - \frac{1}{\alpha}) = 2$; that is $\alpha = \frac{2\,(d+2)}{d}$.

We also have from (2.1.2) and the fifth bound in (3.2.5), for $n = 1, \cdots, N$, that

$$\|V_\varepsilon^n\|_{0,\beta}^{\beta} \leq C\, \|V_\varepsilon^n\|_{0,1}^{\beta-2} \|V_\varepsilon^n\|_1^2 \leq C\, \|V_\varepsilon^n\|_1^2, \tag{3.2.16}$$

where $\beta\,(\frac{2\,d\,(\beta-1)}{\beta\,(d+2)}) = 2$; that is $\beta = \frac{2\,(d+1)}{d}$.

On noting that $\psi_\varepsilon(s) \le |s| + \varepsilon$, it follows from (3.2.16) for $n = 1, \cdots , N$ that

$$\|\psi_\varepsilon(V_\varepsilon^n)\|_{0,\beta}^\beta \le C + \|V_\varepsilon^n\|_{0,\beta}^\beta \le C\, \left(1 + \|V_\varepsilon^n\|_1^2\right) . \tag{3.2.17}$$

After recalling that $\psi_\varepsilon(s) \le [s]_+ + \varepsilon = s - [s]_- + \varepsilon$, we obtain from the Young's inequality and the third and the fifth bounds in (3.2.5) that for $n = 1, \cdots , N$

$$\begin{aligned}\|\pi^h\psi_\varepsilon(V_\varepsilon^n)\|_{0,1} &\le \int_\Omega \left(V_\varepsilon^n - \pi^h[V_\varepsilon^n]_- + \varepsilon\right) \mathrm{d}x \\ &\le C\, \left(1 + \|V_\varepsilon^n\|_{0,1} + \|\pi^h[V_\varepsilon^n]_-\|_0^2\right) \le C\,,\end{aligned} \tag{3.2.18}$$

and

$$\|\pi^h\psi_\varepsilon(V_\varepsilon^n)\|_0^2 \le C\, \left(1 + \|V_\varepsilon^n\|_0^2 + \|\pi^h[V_\varepsilon^n]_-\|_0^2\right) \le C\, \left(1 + \|V_\varepsilon^n\|_0^2\right) . \tag{3.2.19}$$

From (2.1.2), (3.2.18), (3.2.19) and (2.4.7) we have for $n = 1, \cdots , N$ that

$$\|\pi^h\psi_\varepsilon(V_\varepsilon^n)\|_{0,\beta}^\beta \le C\, \|\pi^h\psi_\varepsilon(V_\varepsilon^n)\|_{0,1}^{\beta-2} \|\pi^h\psi_\varepsilon(V_\varepsilon^n)\|_1^2 \le C\, \left(1 + \|V_\varepsilon^n\|_1^2\right) . \tag{3.2.20}$$

Now, on noting (3.2.15) $\rightarrow$ (3.2.20), the bounds $1 \rightarrow 4$ in (3.2.14) follow from (3.2.5). The fifth bound in (3.2.14) follows from the fourth bound in (3.2.14) since we have from (2.4.29), (2.4.30), (2.3.11) and (2.4.15), for $n = 1, \cdots , N$, that

$$\begin{aligned}\|\Xi_\varepsilon(V_\varepsilon^n)\|_{0,\beta}^\beta := \int_\Omega \|\Xi_\varepsilon(V_\varepsilon^n(x))\|^\beta \,\mathrm{d}x &= \sum_{\tau\in\mathcal{T}^h} \int_\tau \|\Xi_\varepsilon(V_\varepsilon^n(x))\|^\beta \,\mathrm{d}x \\ &\le C \sum_{\tau\in\mathcal{T}^h} h_\tau^d\, \|\pi^h\psi_\varepsilon(V_\varepsilon^n)\|_{0,\infty,\tau}^\beta \le C \sum_{\tau\in\mathcal{T}^h} \|\pi^h\psi_\varepsilon(V_\varepsilon^n)\|_{0,\beta,\tau}^\beta \\ &\le C\, \|\pi^h\psi_\varepsilon(V_\varepsilon^n)\|_{0,\beta}^\beta \,.\end{aligned}$$

Before we start discussing the last four bounds in (3.2.14), it is useful to note from the definition of π^h , (3.1.2) and the assumptions on u^0 and v^0 that

$$\|U_\varepsilon^0\|_0 + \|V_\varepsilon^0\|_0 \le C\, \left(\|u^0\|_{0,\infty} + \|v^0\|_{0,\infty}\right) \le C\,. \tag{3.2.21}$$

We now consider the sixth bound in (3.2.14). It follows from (3.1.8), (3.1.1), (2.4.31a), (2.4.2), (2.3.5), (3.1.3), (2.4.24), (2.4.25) and (3.2.19) for any $\eta \in H^1(\Omega)$

and for $n = 1, \cdots, N$ that

$$
\begin{aligned}
\langle \tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t_n}, \eta \rangle = \left(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t_n}, \eta \right) &= \left(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t_n}, P^h \eta \right)^h \\
&= (U_\varepsilon^n - U_\varepsilon^n \, \phi_\varepsilon(U_\varepsilon^{n-1}) - \phi_\varepsilon(U_\varepsilon^n) \, \psi_\varepsilon(V_\varepsilon^{n-1}), P^h \eta)^h \\
&\qquad - \left(D \, \nabla U_\varepsilon^n + \Lambda_\varepsilon(U_\varepsilon^n) \nabla (U_\varepsilon^n + V_\varepsilon^n), \nabla P^h \eta \right) \\
&\leq C \, \left(\|U_\varepsilon^n\|_0 + \|\pi^h \psi_\varepsilon(V_\varepsilon^{n-1})\|_0 \right) \, \|P^h \eta\|_0 \\
&\qquad + C \, \left(|U_\varepsilon^n|_1 + |U_\varepsilon^n + V_\varepsilon^n|_1 \right) \, |P^h \eta|_1 \\
&\leq C \, \left(\|U_\varepsilon^n\|_1 + \|V_\varepsilon^n\|_1 + \|V_\varepsilon^{n-1}\|_0 \right) \, \|\eta\|_1 \,,
\end{aligned} \tag{3.2.22}
$$

and therefore,

$$
\| \tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t_n} \|_{(H^1(\Omega))'}^2 \leq C \, \left(\|U_\varepsilon^n\|_1^2 + \|V_\varepsilon^n\|_1^2 + \|V_\varepsilon^{n-1}\|_0^2 \right) \,. \tag{3.2.23}
$$

Hence, we have from (3.2.23), (3.2.5), our assumption on the time steps and (3.2.21) that

$$
\sum_{n=1}^{N} \Delta t_n \, \| \tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t_n} \|_{(H^1(\Omega))'}^2 \leq C \, \sum_{n=1}^{N} \Delta t_n \, \left(\|U_\varepsilon^n\|_1^2 + \|V_\varepsilon^n\|_1^2 + \|V_\varepsilon^{n-1}\|_0^2 \right) \leq C \,.
$$

To derive the seventh bound in (3.2.14), we first note from the generalized Hölder's inequality (2.1.7) and (3.1.3) for any $\eta \in W^{1,q'}(\Omega)$ and for $n = 1, \cdots, N$ that

$$
\begin{aligned}
\left| \left(\Xi_\varepsilon(V_\varepsilon^n) \nabla (U_\varepsilon^n + V_\varepsilon^n), \nabla P^h \eta \right) \right| &\leq \|\Xi_\varepsilon(V_\varepsilon^n)\|_{0,\beta} |U_\varepsilon^n + V_\varepsilon^n|_1 |P^h \eta|_{1,q'} \\
&\leq C \, \|\Xi_\varepsilon(V_\varepsilon^n)\|_{0,\beta} \|U_\varepsilon^n + V_\varepsilon^n\|_1 \|\eta\|_{1,q'} \,,
\end{aligned} \tag{3.2.24}
$$

where $\frac{1}{2} + \frac{1}{\beta} + \frac{1}{q'} = 1$; that is $q' = 2\,(d+1)$.
On noting (2.4.3), (2.4.2), (3.2.19), (3.1.2) and the embedding $L^\beta(\Omega) \hookrightarrow L^2(\Omega)$, we have for any $\eta \in W^{1,q'}(\Omega)$ and for $n = 1, \cdots, N$ that

$$
\begin{aligned}
\left| \left(\psi_\varepsilon(V_\varepsilon^n) \, \psi_\varepsilon(V_\varepsilon^{n-1}), P^h \eta \right)^h \right| &\leq C \, \|\pi^h \psi_\varepsilon(V_\varepsilon^n)\|_0 \, \|\pi^h \psi_\varepsilon(V_\varepsilon^{n-1})\|_0 \, \|P^h \eta\|_{0,\infty} \\
&\leq C \, \|\pi^h \psi_\varepsilon(V_\varepsilon^n)\|_{0,\beta} \, \left(1 + \|V_\varepsilon^{n-1}\|_0 \right) \, \|\eta\|_{0,\infty} \\
&\leq C \, \|\pi^h \psi_\varepsilon(V_\varepsilon^n)\|_{0,\beta} \, \left(1 + \|V_\varepsilon^{n-1}\|_0 \right) \, \|\eta\|_{1,q'},
\end{aligned} \tag{3.2.25}
$$

where we have also employed, from (2.1.1), the following continuous embedding

$$W^{1,q'}(\Omega) \hookrightarrow L^\infty(\Omega)\,.$$

Similarly to (3.2.22), it follows from (3.1.5), (3.1.1), (2.4.31b), (3.1.3), (3.2.24) and (3.2.25) for any $\eta \in W^{1,q'}(\Omega)$ and for $n = 1, \cdots, N$ that

$$\langle \tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\Delta t_n}, \eta \rangle_{q'} = \left(\tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\Delta t_n}, \eta \right) = \left(\tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\Delta t_n}, P^h \eta \right)^h \leq C\, \mathcal{A}_n\, \mathcal{B}_n\, \|\eta\|_{1,q'}\,, \tag{3.2.26}$$

where

$$\mathcal{A}_n = \|U_\varepsilon^n\|_1 + \|V_\varepsilon^n\|_1 + \|V_\varepsilon^{n-1}\|_0 + 1\,,$$
$$\mathcal{B}_n = \|\pi^h \psi_\varepsilon(V_\varepsilon^n)\|_{0,\beta} + \|\Xi_\varepsilon(V_\varepsilon^n)\|_{0,\beta} + 1\,.$$

Thus, (3.2.26) implies

$$\| \tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\Delta t_n} \|_{(W^{1,q'}(\Omega))'} \leq C\, \mathcal{A}_n\, \mathcal{B}_n\,. \tag{3.2.27}$$

Hence we have from (3.2.27), the Cauchy-Schwarz inequality, (3.2.5), the bounds $(4-5)$ in (3.2.14), the assumption on the temporal discretization and (3.2.21) that

$$\begin{aligned}\sum_{n=1}^N \Delta t_n \| \tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\Delta t_n} \|^q_{(W^{1,q'}(\Omega))'} &\leq C \sum_{n=1}^N \Delta t_n\, \mathcal{A}_n^q\, \mathcal{B}_n^q \\ &\leq C \left[\sum_{n=1}^N \Delta t_n\, \mathcal{A}_n^2 \right]^{\frac{q}{2}} \left[\sum_{n=1}^N \Delta t_n\, \mathcal{B}_n^\beta \right]^{\frac{q}{\beta}} \leq C\,,\end{aligned}$$

where $\frac{1}{2} + \frac{1}{\beta} = \frac{1}{q}$; that is $\frac{1}{q} + \frac{1}{q'} = 1$ and $q = \frac{2\,(d+1)}{2\,d+1}$.

To complete the proof of the theorem, we note that the last two bounds in (3.2.14) follow from the sixth and the seventh bounds in (3.2.14), respectively, on recalling (3.1.9) and (3.1.6) . □

As the condition $u^0\,,\ v^0 \in H^1(\Omega)$ will be essential in the analysis of the next section, we close this section by giving the following short lemma:

Lemma 3.2.4 Let the assumptions (A) hold and let $u^0\,,\ v^0 \in H^1_{\geq 0}(\Omega)$. On choosing either $U_\varepsilon^0 \equiv P^h u^0$ and $V_\varepsilon^0 \equiv P^h v^0$; or $U_\varepsilon^0 \equiv \pi^h u^0$ and $V_\varepsilon^0 \equiv \pi^h v^0$ if either $d = 1$ or $u^0\,,\ v^0 \in W^{1,r}(\Omega)$ with $r > d$, it follows that $U_\varepsilon^0\,,\ V_\varepsilon^0 \in S^h_{\geq 0}$ and

$$\|U_\varepsilon^0\|_1^2 + \|V_\varepsilon^0\|_1^2 \leq C\,. \tag{3.2.28}$$

Proof: We first mention that $\pi^h u^0$ and $\pi^h v^0$ are well defined as the Sobolev embedding result (see Ciarlet [23], page 114):

$$W^{m,r}(\Omega) \overset{c}{\hookrightarrow} C(\overline{\Omega}) \qquad \text{holds for } r \in [1, \infty] \quad \text{if} \quad m > \tfrac{d}{r}\,.$$

It can be seen clearly from the definitions of the projection operator P^h and the interpolation operator π^h that U_ε^0 , $V_\varepsilon^0 \in S^h_{\geq 0}$. The bound (3.2.28) follows immediately on noting (2.4.16), (3.1.3) and the assumptions on u^0 , v^0 . □

3.3 Existence of a weak solution

In this section we prove the global existence of a non-negative weak solution of the continuous problem (2.2.4a)-(2.2.4d) . This is achieved by taking the limit of the regularization and discretization parameters of the problem ($\mathrm{P}^{h,\Delta t}_{M,\varepsilon}$).

We begin by introducing the following definitions:

Let

$$U_\varepsilon(t) := \left(\frac{t - t_{n-1}}{\Delta t_n}\right) U_\varepsilon^n + \left(\frac{t_n - t}{\Delta t_n}\right) U_\varepsilon^{n-1} \qquad t \in [t_{n-1}, t_n] \quad n \geq 1\,, \tag{3.3.1a}$$

$$V_\varepsilon(t) := \left(\frac{t - t_{n-1}}{\Delta t_n}\right) V_\varepsilon^n + \left(\frac{t_n - t}{\Delta t_n}\right) V_\varepsilon^{n-1} \qquad t \in [t_{n-1}, t_n] \quad n \geq 1\,, \tag{3.3.1b}$$

and

$$U_\varepsilon^+(t) := U_\varepsilon^n, \qquad U_\varepsilon^-(t) := U_\varepsilon^{n-1} \qquad t \in (t_{n-1}, t_n] \quad n \geq 1\,, \tag{3.3.2a}$$

$$V_\varepsilon^+(t) := V_\varepsilon^n, \qquad V_\varepsilon^-(t) := V_\varepsilon^{n-1} \qquad t \in (t_{n-1}, t_n] \quad n \geq 1\,. \tag{3.3.2b}$$

On noting (3.3.1a,b) and (3.3.2a,b) we have that

$$\frac{\partial U_\varepsilon}{\partial t} = \frac{U_\varepsilon^+ - U_\varepsilon^-}{\Delta t_n} = \frac{U_\varepsilon^+ - U_\varepsilon}{t_n - t} = \frac{U_\varepsilon - U_\varepsilon^-}{t - t_{n-1}} \qquad t \in (t_{n-1}, t_n) \quad n \geq 1\,, \tag{3.3.3a}$$

$$\frac{\partial V_\varepsilon}{\partial t} = \frac{V_\varepsilon^+ - V_\varepsilon^-}{\Delta t_n} = \frac{V_\varepsilon^+ - V_\varepsilon}{t_n - t} = \frac{V_\varepsilon - V_\varepsilon^-}{t - t_{n-1}} \qquad t \in (t_{n-1}, t_n) \quad n \geq 1\,. \tag{3.3.3b}$$

Using the above notation, the problem $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ can be restated as follows:
Find $\{U_\varepsilon,\, V_\varepsilon\} \in C([0,T];S^h) \times C([0,T];S^h)$ such that for all $\chi \in L^2(0,T;S^h)$

$$\int_0^T \Big[\big(\tfrac{\partial U_\varepsilon}{\partial t}, \chi\big)^h + D\,(\,\nabla U_\varepsilon^+, \nabla\chi) + (\,\Lambda_\varepsilon(U_\varepsilon^+)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla\chi)\Big]\;\mathrm{d}t$$
$$= \int_0^T \big(\,U_\varepsilon^+ - U_\varepsilon^+\,\phi_\varepsilon(U_\varepsilon^-) - \phi_\varepsilon(U_\varepsilon^+)\,\psi_\varepsilon(V_\varepsilon^-), \chi\big)^h\;\mathrm{d}t\,, \tag{3.3.4a}$$
$$\int_0^T \Big[\big(\tfrac{\partial V_\varepsilon}{\partial t}, \chi\big)^h + D\,(\,\nabla V_\varepsilon^+, \nabla\chi) + (\,\Xi_\varepsilon(V_\varepsilon^+)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla\chi)\Big]\;\mathrm{d}t$$
$$= \int_0^T \big(\,\gamma\,V_\varepsilon^+ - \psi_\varepsilon(V_\varepsilon^+)\;\big[\phi_\varepsilon(U_\varepsilon^-) + \psi_\varepsilon(V_\varepsilon^-)\big]\,, \chi\big)^h\;\mathrm{d}t\,. \tag{3.3.4b}$$

The argument in this section will consist of three main steps. We first utilize the stability estimates derived in Section 3.2 on the approximate solutions. Then we prove the existence of non-negative functions $\{u,v\}$ bounded in various time-dependent spaces using a classical sequential compactness arguments (see the results collected in A.1.5 $\to$ A.1.10). Finally, we prove that the functions $\{u,v\}$ represent a global weak solution of the system (2.2.4a)-(2.2.4d) via passage to the limit $\varepsilon\,,\ h\,,\ \Delta t \to 0$ of the approximate system (3.3.4a)-(3.3.4b) .

Theorem 3.3.1 Let the assumptions (A) hold, $D > 0\,,\ \gamma > 1$ and $u^0\,,\ v^0 \in H^1_{\geq 0}(\Omega) \cap L^\infty(\Omega)$. In addition, let $\big\{\,\varepsilon, h, \{\Delta t_n\}_{n=1}^N, U_\varepsilon^0, V_\varepsilon^0\,\big\}$ be such that

(i) either $U_\varepsilon^0 \equiv P^h u^0$ and $V_\varepsilon^0 \equiv P^h v^0$; or $U_\varepsilon^0 \equiv \pi^h u^0$ and $V_\varepsilon^0 \equiv \pi^h v^0$ if either $d = 1$ or $u^0\,,\ v^0 \in W^{1,r}(\Omega)$ with $r > d$.

(ii) $\Delta t \leq \frac{1-\delta}{2\,\gamma+2}$, for some $\delta \in (0,1)$.

(iii) $\Delta t_n \leq C\,\Delta t_{n-1} \quad \forall\ n = 2, \cdots, N$.

(iv) $\Delta t\ ,\ \varepsilon\ \to 0 \quad$ as $h\ \to 0$.

Then there exists a subsequence of $\{U_\varepsilon, V_\varepsilon\}_{h>0}$, solving (3.3.4a)-(3.3.4b) , and functions

$$u \in L^2(0,T;H^1(\Omega)) \cap L^\alpha(\Omega_T) \cap L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;(H^1(\Omega))')\,, \tag{3.3.5a}$$
$$v \in L^2(0,T;H^1(\Omega)) \cap L^\beta(\Omega_T) \cap W^{1,q}(0,T;(W^{1,q'}(\Omega))')\,, \tag{3.3.5b}$$

with $u(x,t)\,,\ v(x,t) \geq 0$ almost everywhere and

$$u(\cdot,0) = u^0(\cdot) \;\text{ in } L^2(\Omega) \quad\text{and}\quad v(\cdot,0) = v^0(\cdot) \;\text{ in } (W^{1,q'}(\Omega))'\,, \tag{3.3.5c}$$

where

$$\alpha = \tfrac{2\,(d+2)}{d}\;,\quad \beta = \tfrac{2\,(d+1)}{d}\;,\quad q = \tfrac{2\,(d+1)}{2\,d+1} \;\text{ and }\; q' = \tfrac{q}{q-1} = 2\,(d+1)\,.$$

Moreover, it holds as $h \to 0$ that

$$U_\varepsilon,\ U_\varepsilon^\pm \rightharpoonup u \qquad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\alpha(\Omega_T)\,, \tag{3.3.6a}$$

$$U_\varepsilon,\ U_\varepsilon^\pm \rightharpoonup^* u \qquad \text{in } L^\infty(0,T;L^2(\Omega))\,, \tag{3.3.6b}$$

$$\frac{\partial U_\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \qquad \text{in } L^2(0,T;(H^1(\Omega))')\,, \tag{3.3.6c}$$

$$U_\varepsilon,\ U_\varepsilon^\pm \to u \qquad \text{in } L^2(0,T;L^s(\Omega))\,, \tag{3.3.6d}$$

$$\phi_\varepsilon(U_\varepsilon^\pm) \to \phi(u) \qquad \text{in } L^2(0,T;L^s(\Omega))\,, \tag{3.3.6e}$$

$$\pi^h\phi_\varepsilon(U_\varepsilon^\pm) \to \phi(u) \qquad \text{in } L^2(0,T;L^s(\Omega))\,, \tag{3.3.6f}$$

$$\Lambda_\varepsilon(U_\varepsilon^\pm) \to \phi(u)\,\mathcal{I} \qquad \text{in } L^2(0,T;L^s(\Omega))\,, \tag{3.3.6g}$$

and

$$V_\varepsilon,\ V_\varepsilon^\pm \rightharpoonup v \qquad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\beta(\Omega_T)\,, \tag{3.3.7a}$$

$$\frac{\partial V_\varepsilon}{\partial t} \rightharpoonup \frac{\partial v}{\partial t} \qquad \text{in } L^q(0,T;(W^{1,q'}(\Omega))')\,, \tag{3.3.7b}$$

$$V_\varepsilon,\ V_\varepsilon^\pm \to v \qquad \text{in } L^2(0,T;L^s(\Omega))\,, \tag{3.3.7c}$$

$$\psi_\varepsilon(V_\varepsilon^\pm) \to v \qquad \text{in } L^2(0,T;L^s(\Omega))\,, \tag{3.3.7d}$$

$$\pi^h\psi_\varepsilon(V_\varepsilon^\pm) \to v \qquad \text{in } L^2(0,T;L^s(\Omega))\,, \tag{3.3.7e}$$

$$\Xi_\varepsilon(V_\varepsilon^\pm) \to v\,\mathcal{I} \qquad \text{in } L^2(0,T;L^s(\Omega))\,, \tag{3.3.7f}$$

for any

$$s \in \begin{cases} [2,\infty] & \text{if } d = 1\,,\\ [2,\infty) & \text{if } d = 2\,,\\ [2,6) & \text{if } d = 3\,; \end{cases}$$

where the symbols "$\to$" "$\rightharpoonup$" and "$\rightharpoonup^*$" represent strong, weak and weak-star convergence respectively (see A.1.5 $\to$ A.1.7).

Proof: By using the assumptions (i)$\rightarrow$(iii), (3.2.5), (3.2.14), (2.3.5), (2.4.24), (2.4.25), (3.3.1a,b), (3.3.2a,b), (3.3.3a,b) and (3.2.28) we obtain the following uniform bounds independently of the parameters ε, h and Δt

$$\begin{aligned}
&\|U_\varepsilon^{(\pm)}\|_{L^2(0,T;H^1(\Omega))} + \|U_\varepsilon^{(\pm)}\|_{L^\alpha(\Omega_T)} + \|U_\varepsilon^{(\pm)}\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad + \varepsilon^{-\frac{1}{2}}\|\pi^h[U_\varepsilon^{(\pm)}]_-\|_{L^\infty(0,T;L^2(\Omega))} + \|\tfrac{\partial U_\varepsilon}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} \\
&\qquad + \|\mathcal{G}\tfrac{\partial U_\varepsilon}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\phi_\varepsilon(U_\varepsilon^{(\pm)})\|_{L^\infty(\Omega_T)} \\
&\qquad\quad + \|\pi^h\phi_\varepsilon(U_\varepsilon^{(\pm)})\|_{L^\infty(\Omega_T)} + \|\Lambda_\varepsilon(U_\varepsilon^{(\pm)})\|_{L^\infty(\Omega_T)} \le C\,,
\end{aligned} \tag{3.3.8a}$$

and

$$\begin{aligned}
&\|V_\varepsilon^{(\pm)}\|_{L^2(0,T;H^1(\Omega))} + \|V_\varepsilon^{(\pm)}\|_{L^\beta(\Omega_T)} + \|V_\varepsilon^{(\pm)}\|_{L^\infty(0,T;L^1(\Omega))} \\
&\quad + \varepsilon^{-\frac{1}{2}}\|\pi^h[V_\varepsilon^{(\pm)}]_-\|_{L^\infty(0,T;L^2(\Omega))} + \|\tfrac{\partial V_\varepsilon}{\partial t}\|_{L^q(0,T;(W^{1,q'}(\Omega))')} \\
&\qquad + \|\mathcal{G}_q\tfrac{\partial V_\varepsilon}{\partial t}\|_{L^q(0,T;W^{1,q}(\Omega))} + \|\psi_\varepsilon(V_\varepsilon^{(\pm)})\|_{L^\beta(\Omega_T)} \\
&\qquad\quad + \|\pi^h\psi_\varepsilon(V_\varepsilon^{(\pm)})\|_{L^\beta(\Omega_T)} + \|\Xi_\varepsilon(V_\varepsilon^{(\pm)})\|_{L^\beta(\Omega_T)} \le C\,,
\end{aligned} \tag{3.3.8b}$$

where $(\pm)$ is an adopted abbreviation for "with" and "without" the superscripts "+" and "−".

Also, we note from (3.3.3a) and the fifth bound in (3.3.8a) that

$$\begin{aligned}
\|U_\varepsilon^\pm - U_\varepsilon\|^2_{L^2(0,T;(H^1(\Omega))')} &= \int_0^T \|U_\varepsilon^\pm - U_\varepsilon\|^2_{(H^1(\Omega))'}\,\mathrm{d}t = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|U_\varepsilon^\pm - U_\varepsilon\|^2_{(H^1(\Omega))'}\,\mathrm{d}t \\
&= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left|t - t_n^\pm\right|^2 \|\tfrac{\partial U_\varepsilon}{\partial t}\|^2_{(H^1(\Omega))'}\,\mathrm{d}t \qquad t_n^+ = t_n\ ,\ t_n^- = t_{n-1} \\
&\le \sum_{n=1}^N (\Delta t_n)^2 \int_{t_{n-1}}^{t_n} \|\tfrac{\partial U_\varepsilon}{\partial t}\|^2_{(H^1(\Omega))'}\,\mathrm{d}t \\
&\le (\Delta t)^2 \int_0^T \|\tfrac{\partial U_\varepsilon}{\partial t}\|^2_{(H^1(\Omega))'}\,\mathrm{d}t \\
&= (\Delta t)^2\,\|\tfrac{\partial U_\varepsilon}{\partial t}\|^2_{L^2(0,T;(H^1(\Omega))')} \le C\,(\Delta t)^2\,.
\end{aligned} \tag{3.3.9a}$$

Similarly to (3.3.9a), we have from (3.3.3b) and the fifth bound in (3.3.8b) that

$$\|V_\varepsilon^\pm - V_\varepsilon\|^q_{L^q(0,T;(W^{1,q'}(\Omega))')} \le (\Delta t)^q\,\|\tfrac{\partial V_\varepsilon}{\partial t}\|^q_{L^q(0,T;(W^{1,q'}(\Omega))')} \le C\,(\Delta t)^q\,. \tag{3.3.9b}$$

We now recall that $L^2(0,T;H^1(\Omega))$ and $L^\alpha(\Omega_T)$ are reflexive Banach spaces, while $L^1(0,T;L^2(\Omega))$, which is the pre-dual[2] of $L^\infty(0,T;L^2(\Omega))$, is separable Banach space but not reflexive (see A.1.12 and A.1.14). Noting that and the first three bounds in (3.3.8a), we deduce from classical compactness arguments the existence of a subsequence $\{U_\varepsilon\}_h$ and a function $u \in L^2(0,T;H^1(\Omega)) \cap L^\alpha(\Omega_T) \cap L^\infty(0,T;L^2(\Omega))$ satisfying the convergence results (3.3.6a)-(3.3.6b). Where we have noticed from (3.3.9a) that the subsequences $\{U_\varepsilon^+ \,,\, U_\varepsilon^- \,,\, U_\varepsilon\}_h$ have the same limit, after recalling that weak and weak-star limits are unique (see A.1.8).

As $L^2(0,T;(H^1(\Omega))')$ is reflexive Banach space, it follows from the fifth bound in (3.3.8a), on employing weak compactness arguments, that there exists $\tilde{\eta} \in L^2(0,T;(H^1(\Omega))')$ such that

$$\frac{\partial U_\varepsilon}{\partial t} \rightharpoonup \tilde{\eta} \qquad \text{in} \;\; L^2(0,T;(H^1(\Omega))')\,.$$

A well known argument can be easily adapted to show that $\tilde{\eta} = \frac{\partial u}{\partial t}$, (see Robinson [58], page 204). Thus, the result (3.3.6c) holds. The result (3.3.5a) follows immediately on noting the embedding $L^2(0,T;H^1(\Omega)) \hookrightarrow L^2(0,T;(H^1(\Omega))')$ since

$$u \in L^2(0,T;H^1(\Omega)) \cap L^\alpha(\Omega_T) \cap L^\infty(0,T;L^2(\Omega)) \quad \text{and} \quad \tfrac{\partial u}{\partial t} \in L^2(0,T;(H^1(\Omega))')\,.$$

From application of the Lions-Aubin theorem, see (2.1.4), on noting the following embedding results

$$H^1(\Omega) \overset{c}{\hookrightarrow} L^s(\Omega) \hookrightarrow (H^1(\Omega))'\,,$$

which hold from the Rellich-Kondrachov theorem under the stated choice of s, we find that

$$W_u = \left\{\eta : \; \eta \in L^2(0,T;H^1(\Omega)),\; \tfrac{\partial \eta}{\partial t} \in L^2(0,T;(H^1(\Omega))')\right\} \overset{c}{\hookrightarrow} L^2(0,T;L^s(\Omega))\,.$$

As $U_\varepsilon \in W_u$, we can extract a subsequence, still denoted U_ε, such that the convergence result (3.3.6d) holds.

[2]Let X , Y are Banach spaces. We say that X is pre-dual of Y if $X' = Y$.

Using the strong convergence of U_ε to u in $L^2(0,T;L^s(\Omega))$ and the fourth bound in (3.3.8a), we can extract a subsequence, still denoted U_ε, such that as $h \to 0$ (see Appendix A.1.11)

$$U_\varepsilon \to u \quad \text{and} \quad \pi^h[U_\varepsilon]_- \to 0 \qquad a.e. \text{ in } \Omega_T. \tag{3.3.10}$$

But we have from the definition of π^h that

$$U_\varepsilon = \pi^h[U_\varepsilon]_+ + \pi^h[U_\varepsilon]_- \,. \tag{3.3.11}$$

Therefore, we deduce from (3.3.10) and (3.3.11) that $u \geq 0$ almost everywhere.

In order to show (3.3.6e) and (3.3.6f), we first note that

$$\left\|\phi_\varepsilon(U_\varepsilon^\pm) - \phi(u)\right\|_{L^2(0,T;L^s(\Omega))} \leq \left\|\phi_\varepsilon(U_\varepsilon^\pm) - \phi_\varepsilon(u)\right\|_{L^2(0,T;L^s(\Omega))} + \|\phi_\varepsilon(u) - \phi(u)\|_{L^2(0,T;L^s(\Omega))} \,. \tag{3.3.12}$$

Noting (2.2.1), (2.3.5), the non-negativity of the function u and the assumption (iv) yields that

$$\|\phi_\varepsilon(u) - \phi(u)\|_{L^2(0,T;L^s(\Omega))} \leq C\,\varepsilon \to 0 \ \text{ as } \ h \to 0\,. \tag{3.3.13}$$

From the Lipschitz continuity of the function ϕ_ε and (3.3.6d), it follows that

$$\left\|\phi_\varepsilon(U_\varepsilon^\pm) - \phi_\varepsilon(u)\right\|_{L^2(0,T;L^s(\Omega))} \leq \left\|U_\varepsilon^\pm - u\right\|_{L^2(0,T;L^s(\Omega))} \to 0 \ \text{ as } \ h \to 0\,. \tag{3.3.14}$$

We also have from (2.4.16), (2.3.5), (2.4.15) and the first bound in (3.3.8a) that

$$\begin{aligned}\left\|(I - \pi^h)\phi_\varepsilon(U_\varepsilon^\pm)\right\|_{L^2(0,T;L^s(\Omega))} &\leq C\,h\,\left\|\nabla U_\varepsilon^\pm\right\|_{L^2(0,T;L^s(\Omega))} \\ &\leq C\,h^{1-d\,(\frac{1}{2}-\frac{1}{s})}\,\left\|U_\varepsilon^\pm\right\|_{L^2(0,T;H^1(\Omega))} \\ &\leq C\,h^{1-d\,(\frac{1}{2}-\frac{1}{s})} \to 0 \ \text{ as } \ h \to 0\,.\end{aligned} \tag{3.3.15}$$

Thus, the results (3.3.6e) and (3.3.6f) follow by combining (3.3.12)→(3.3.15).

We obtain from (2.4.46), (2.4.15), the first bound in (3.3.8a) and (3.3.6e) that

$$\begin{aligned}&\left\|\Lambda_\varepsilon(U_\varepsilon^\pm) - \phi(u)\,\mathcal{I}\right\|_{L^2(0,T;L^s(\Omega))} \\ &\quad\leq \left\|\Lambda_\varepsilon(U_\varepsilon^\pm) - \phi_\varepsilon(U_\varepsilon^\pm)\,\mathcal{I}\right\|_{L^2(0,T;L^s(\Omega))} + \left\|\phi_\varepsilon(U_\varepsilon^\pm) - \phi(u)\right\|_{L^2(0,T;L^s(\Omega))} \\ &\quad\leq C\,h^{1-d\,(\frac{1}{2}-\frac{1}{s})}\,\left\|U_\varepsilon^\pm\right\|_{L^2(0,T;H^1(\Omega))} + \left\|\phi_\varepsilon(U_\varepsilon^\pm) - \phi(u)\right\|_{L^2(0,T;L^s(\Omega))} \\ &\quad\leq C\,h^{1-d\,(\frac{1}{2}-\frac{1}{s})} + \left\|\phi_\varepsilon(U_\varepsilon^\pm) - \phi(u)\right\|_{L^2(0,T;L^s(\Omega))} \to 0 \ \text{ as } \ h \to 0\,.\end{aligned} \tag{3.3.16}$$

Hence the result (3.3.6g) holds from (3.3.16).

Similarly to $\{U_\varepsilon^{(\pm)}\}$, the convergence results for $\{V_\varepsilon^{(\pm)}\}$ in (3.3.7a)-(3.3.7c) follow from classical compactness arguments on noting the bounds (1, 2 and 5) in (3.3.8b), (3.3.9b) and the following application of (2.1.4):

$$W_v = \left\{\eta :\ \eta \in L^2(0,T;H^1(\Omega)),\ \tfrac{\partial \eta}{\partial t} \in L^q(0,T;(W^{1,q'}(\Omega))')\right\} \overset{c}{\hookrightarrow} L^2(0,T;L^s(\Omega))\,,$$

after recalling that $L^2(0,T;H^1(\Omega))$, $L^\beta(\Omega_T)$ and $L^q(0,T;(W^{1,q'}(\Omega))')$ are reflexive Banach spaces. As a result, we have

$$v \in L^2(0,T;H^1(\Omega)) \cap L^\beta(\Omega_T) \quad \text{and} \quad \tfrac{\partial v}{\partial t} \in L^q(0,T;(W^{1,q'}(\Omega))')\,.$$

Noting this and the embedding $L^2(0,T;H^1(\Omega)) \hookrightarrow L^q(0,T;(W^{1,q'}(\Omega))')$ gives (3.3.5b). The fourth bound in (3.3.8b) and (3.3.7c) implies that $v \geq 0$ almost everywhere. We now show the results (3.3.7d)-(3.3.7f) by adapting the arguments used for deriving (3.3.6e)-(3.3.6g). First we note that

$$\left\|\psi_\varepsilon(V_\varepsilon^\pm) - v\right\|_{L^2(0,T;L^s(\Omega))} \leq \left\|\psi_\varepsilon(V_\varepsilon^\pm) - \psi_\varepsilon(v)\right\|_{L^2(0,T;L^s(\Omega))} + \|\psi_\varepsilon(v) - v\|_{L^2(0,T;L^s(\Omega))}\,. \tag{3.3.17}$$

After recalling that $v \in L^2(0,T;H^1(\Omega)) \hookrightarrow L^2(0,T;L^s(\Omega))$, we obtain from (2.3.11), the non-negativity of the function v and the assumption (iv), on using the dominated convergence theorem, that

$$\|\psi_\varepsilon(v) - v\|_{L^2(0,T;L^s(\Omega))} \to 0 \quad \text{as} \quad h \to 0\,. \tag{3.3.18}$$

Also, we have from the Lipschitz continuity of the function ψ_ε and (3.3.7c) that

$$\left\|\psi_\varepsilon(V_\varepsilon^\pm) - \psi_\varepsilon(v)\right\|_{L^2(0,T;L^s(\Omega))} \leq \left\|V_\varepsilon^\pm - v\right\|_{L^2(0,T;L^s(\Omega))} \to 0 \ \text{ as } \ h \to 0\,. \tag{3.3.19}$$

Similarly to (3.3.15) and (3.3.16), it follows from (2.4.16), (2.3.11), (2.4.15), (2.4.47), the first bound in (3.3.8b) that

$$\begin{aligned}\left\|(I - \pi^h)\psi_\varepsilon(V_\varepsilon^\pm)\right\|_{L^2(0,T;L^s(\Omega))} &+ \left\|\Xi_\varepsilon(V_\varepsilon^\pm) - v\,\mathcal{I}\right\|_{L^2(0,T;L^s(\Omega))} \\ &\leq C\, h^{1-d\,(\frac{1}{2}-\frac{1}{s})} + \left\|\psi_\varepsilon(V_\varepsilon^\pm) - v\right\|_{L^2(0,T;L^s(\Omega))}\,.\end{aligned} \tag{3.3.20}$$

Thereby, the convergence results (3.3.7d)-(3.3.7f) follow by combining (3.3.17), (3.3.18), (3.3.19) and (3.3.20).

To complete the proof of the theorem, we still have to deal with the initial approximations and show that the solution $\{u, v\}$ satisfies (3.3.5c). We first note from the error estimates (3.1.3) and (2.4.16) and the stated assumptions on the initial data, u^0 and v^0, that

$$\left\|u^0 - P^h u^0\right\|_0 \leq C\,h\,\left|u^0\right|_1 \leq C\,h\,,$$
$$\left\|v^0 - P^h v^0\right\|_0 \leq C\,h\,\left|v^0\right|_1 \leq C\,h\,,$$

and

$$\left\|u^0 - \pi^h u^0\right\|_0 \leq \begin{cases} C\,h\,|u^0|_1 \leq C\,h & \text{for } d = 1\,, \\ C\,h\,|u^0|_{1,r} \leq C\,h & \text{for } d = 2 \text{ or } 3\,, \end{cases}$$
$$\left\|v^0 - \pi^h v^0\right\|_0 \leq \begin{cases} C\,h\,|v^0|_1 \leq C\,h & \text{for } d = 1\,, \\ C\,h\,|v^0|_{1,r} \leq C\,h & \text{for } d = 2 \text{ or } 3\,, \end{cases}$$

which provide the following strong convergence results as $h \to 0$

$$U_\varepsilon^0 \longrightarrow u^0 \quad \text{in } L^2(\Omega), \tag{3.3.21a}$$
$$V_\varepsilon^0 \longrightarrow v^0 \quad \text{in } L^2(\Omega). \tag{3.3.21b}$$

It follows from (3.3.6d), (3.3.7c) that for *a.e.* (see Theorem A.1.11)

$$U_\varepsilon(t) \longrightarrow u(t) \quad \text{in } L^2(\Omega) \quad \text{as } \varepsilon \to 0\,, \tag{3.3.22a}$$
$$V_\varepsilon(t) \longrightarrow v(t) \quad \text{in } L^2(\Omega) \quad \text{as } \varepsilon \to 0\,, \tag{3.3.22b}$$

We comment that (3.3.21a,b) and (3.3.22a,b) are not sufficient to prove the equalities in (3.3.5c) since if $t = 0$ belongs to the null-set of the almost everywhere statement for (3.3.22a,b) then possibly $u(0) \neq u^0$, $v(0) \neq v^0$ (see Robinson [58], Section 7.4.4, for further discussion). In addition to (3.3.21a,b) and (3.3.22a,b), we actually exploit other properties of the solutions $\{U_\varepsilon, V_\varepsilon\}$ and the functions $\{u, v\}$ in order to conclude that (3.3.5c) holds.

We note that since

$$U_\varepsilon\,,\ u \in L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \tfrac{\partial U_\varepsilon}{\partial t}\,,\ \tfrac{\partial u}{\partial t} \in L^2(0,T;(H^1(\Omega))')\,,$$

and

$$V_\varepsilon\,,\ v \in W^{1,q}(0,T;(W^{1,q'}(\Omega))')\,,$$

it follows that

$$U_\varepsilon\,,\ u \in C([0,T];L^2(\Omega))\,, \tag{3.3.23a}$$

$$V_\varepsilon\,,\ v \in C([0,T];(W^{1,q'}(\Omega))')\,; \tag{3.3.23b}$$

see Theorem 7.2 and Proposition 7.1 in Robinson [58], respectively.

Therefore, the desired result (3.3.5c) follows easily by combining (3.3.21a,b), (3.3.22a,b) and (3.3.23a,b). This ends the proof of the theorem. $\square$

In the following lemma we prove further convergence results which are required for studying the convergence of the system (3.3.4a)-(3.3.4b) in Theorem 3.3.3.

Lemma 3.3.2 Let the assumptions of Theorem 3.3.1 hold. Then the following convergence results are valid as $h \to 0$:

$$U_\varepsilon^+\,\pi^h\phi_\varepsilon(U_\varepsilon^-) \to u\,\phi(u) \qquad \text{in} \quad L^q(\Omega_T)\,, \tag{3.3.24a}$$

$$\pi^h\psi_\varepsilon(V_\varepsilon^\pm)\,\pi^h\phi_\varepsilon(U_\varepsilon^\pm) \to v\,\phi(u) \qquad \text{in} \quad L^q(\Omega_T)\,, \tag{3.3.24b}$$

$$\pi^h\psi_\varepsilon(V_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-) \to v^2 \qquad \text{in} \quad L^q(\Omega_T)\,, \tag{3.3.24c}$$

where $q = \frac{2\,(d+1)}{2\,d+1}$.

Proof: On noting (2.2.1), (3.3.8a) and the embedding $L^\alpha(\Omega_T) \hookrightarrow L^\beta(\Omega_T)$ we have from the Hölder's inequality that

$$\begin{aligned}
&\left\|U_\varepsilon^+\,\pi^h\phi_\varepsilon(U_\varepsilon^-) - u\,\phi(u)\right\|_{L^q(\Omega_T)} \\
&\quad \leq \left\|\pi^h\phi_\varepsilon(U_\varepsilon^-) - \phi(u)\right\|_{L^2(\Omega_T)} \left\|U_\varepsilon^+\right\|_{L^\beta(\Omega_T)} + \left\|U_\varepsilon^+ - u\right\|_{L^q(\Omega_T)} \left\|\phi(u)\right\|_{L^\infty(\Omega_T)} \\
&\quad \leq C\,\left(\left\|\pi^h\phi_\varepsilon(U_\varepsilon^-) - \phi(u)\right\|_{L^2(\Omega_T)} + \left\|U_\varepsilon^+ - u\right\|_{L^2(\Omega_T)}\right)\,.
\end{aligned} \tag{3.3.25}$$

Similarly to (3.3.25), it follows from the Hölder's inequality, (2.3.5) and (3.3.5b), on noting the Sobolev embedding $L^2(\Omega_T) \hookrightarrow L^q(\Omega_T)$, that

$$\begin{aligned}
&\left\|\pi^h\psi_\varepsilon(V_\varepsilon^\pm)\,\pi^h\phi_\varepsilon(U_\varepsilon^\pm) - v\,\phi(u)\right\|_{L^q(\Omega_T)} \\
&\quad\le \left\|\pi^h\psi_\varepsilon(V_\varepsilon^\pm) - v\right\|_{L^q(\Omega_T)} \left\|\pi^h\phi_\varepsilon(U_\varepsilon^\pm)\right\|_{L^\infty(\Omega_T)} + \left\|\pi^h\phi_\varepsilon(U_\varepsilon^\pm) - \phi(u)\right\|_{L^2(\Omega_T)} \|v\|_{L^\beta(\Omega_T)} \\
&\quad\le C\left(\left\|\pi^h\psi_\varepsilon(V_\varepsilon^\pm) - v\right\|_{L^2(\Omega_T)} + \left\|\pi^h\phi_\varepsilon(U_\varepsilon^\pm) - \phi(u)\right\|_{L^2(\Omega_T)}\right). \qquad (3.3.26)
\end{aligned}$$

With the aid of the Hölder's inequality, we also obtain from (3.3.8b) and (3.3.5b) that

$$\begin{aligned}
&\left\|\pi^h\psi_\varepsilon(V_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-) - v^2\right\|_{L^q(\Omega_T)} \\
&\quad\le \left\|\pi^h\psi_\varepsilon(V_\varepsilon^-) - v\right\|_{L^2(\Omega_T)} \left\|\pi^h\psi_\varepsilon(V_\varepsilon^+)\right\|_{L^\beta(\Omega_T)} + \left\|\pi^h\psi_\varepsilon(V_\varepsilon^+) - v\right\|_{L^2(\Omega_T)} \|v\|_{L^\beta(\Omega_T)} \\
&\quad\le C\left(\left\|\pi^h\psi_\varepsilon(V_\varepsilon^-) - v\right\|_{L^2(\Omega_T)} + \left\|\pi^h\psi_\varepsilon(V_\varepsilon^+) - v\right\|_{L^2(\Omega_T)}\right). \qquad (3.3.27)
\end{aligned}$$

The desired results (3.3.24a)-(3.3.24c) follow from (3.3.25), (3.3.26), (3.3.27), (3.3.6d), (3.3.6e) and (3.3.7e) on noting the embedding $L^2(0,T;L^s(\Omega)) \hookrightarrow L^2(\Omega_T)$. □

We are now in the position to prove that the functions $\{u, v\}$, generated from Theorem 3.3.1, represent a global weak solution of problem (P$_M$). We do this by analysing the convergence of the approximate system (3.3.4a)-(3.3.4b). We remark that our proof of the convergence of (3.3.4a) will require us to define $\pi^h\eta$ for a test function $\eta \in L^2(0,T;H^1(\Omega))$. Obviously, $\pi^h\eta$ is well defined in the case $d = 1$, as $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$, but not necessarily for $d = 2$ and 3. However, with the exception of defining $\pi^h\eta$ and using (2.4.16) and (2.4.17) the proof only requires that $\eta \in L^2(0,T;H^1(\Omega))$. Fortunately, we can overcome this obstacle by proving convergence for all $\eta \in L^2(0,T;W^{1,d+1}(\Omega))$ and using the denseness of $L^2(0,T;W^{1,d+1}(\Omega))$ in $L^2(0,T;H^1(\Omega))$ to conclude the convergence for any $\eta \in L^2(0,T;H^1(\Omega))$. In other words, it will be sufficient to prove convergence for all $\eta \in L^2(0,T;H^1(\Omega))$ while assuming the validity of the definition $\pi^h\eta$ and the estimates (2.4.16) and (2.4.17). With this in mind, and for ease of exposition, we write the proof starting with $\eta \in L^2(0,T;H^1(\Omega))$.

Theorem 3.3.3 Let the assumptions of Theorem 3.3.1 hold. Then there exists a subsequence of $\{U_\varepsilon, V_\varepsilon\}_{h>0}$, where $\{U_\varepsilon, V_\varepsilon\}$ solves (3.3.4a)-(3.3.4b), and non-negative functions $\{u, v\}$ satisfying (3.3.5a)-(3.3.5c). In addition, as $h \to 0$ the convergence results (3.3.6a)-(3.3.6g), (3.3.7a)-(3.3.7f) and (3.3.24a)-(3.3.24c) hold. Furthermore, the functions $\{u, v\}$ represent a global weak solution of the problem (P_M) in sense that

$$\int_0^T \left[\left\langle \tfrac{\partial u}{\partial t}, \eta\right\rangle + D\,(\,\nabla u, \nabla \eta) + (\,\phi(u)\,\nabla(u+v), \nabla\eta)\right]\,\mathrm{d}t$$
$$= \int_0^T (\,u - \phi(u)\ [\,u+v\,]\,, \eta)\ \mathrm{d}t \qquad \forall \eta \in L^2(0,T;H^1(\Omega)) \tag{3.3.28a}$$

and

$$\int_0^T \left[\left\langle \tfrac{\partial v}{\partial t}, \eta\right\rangle_{q'} + D\,(\,\nabla v, \nabla \eta) + (\,v\,\nabla(u+v), \nabla\eta)\right]\,\mathrm{d}t$$
$$= \int_0^T (\gamma\, v - v\ [\,\phi(u) + v\,]\,, \eta)\ \mathrm{d}t \qquad \forall \eta \in L^{q'}(0,T;W^{1,q'}(\Omega))\,, \tag{3.3.28b}$$

where $q' = 2\,(d+1)$.

Proof: The first and second parts of the theorem follow from Theorem 3.3.1 and Lemma 3.3.2. To show that $\{u(x,t), v(x,t)\}$ is a weak solution of (P_M) in sense that (3.3.28a)-(3.3.28b) are satisfied, we set $\chi \equiv \pi^h \eta$ as a test function in (3.3.4a)-(3.3.4b) and then pass to the limit ε, h, $\Delta t \to 0$. We first show (3.3.28a) and then we prove (3.3.28b) by following a similar argument.

For any $\eta \in L^2(0,T;H^1(\Omega))$, we set $\chi \equiv \pi^h \eta$ as a test function in (3.3.4a) yielding

$$\int_0^T \left[\left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h \eta\right)^h + D\,(\,\nabla U_\varepsilon^+, \nabla \pi^h \eta) + (\,\Lambda_\varepsilon(U_\varepsilon^+)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla\pi^h\eta)\right]\,\mathrm{d}t$$
$$= \int_0^T \left(\,U_\varepsilon^+ - U_\varepsilon^+\,\phi_\varepsilon(U_\varepsilon^-) - \phi_\varepsilon(U_\varepsilon^+)\,\psi_\varepsilon(V_\varepsilon^-), \pi^h\eta\right)^h\,\mathrm{d}t\,. \tag{3.3.29}$$

We shall now study the convergence of each term in (3.3.29) separately.

For all $\eta \in L^2(0,T;H^1(\Omega))$ and for all $\tilde{\eta} \in H^1(0,T;H^1(\Omega))$ we have that

$$\begin{aligned}
\int_0^T \left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h \eta\right)^h \,\mathrm{d}t = & \int_0^T \left[\left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h[\eta - \tilde{\eta}]\right)^h - \left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h[\eta - \tilde{\eta}]\right)\right] \mathrm{d}t \\
& + \int_0^T \left[\left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h \tilde{\eta}\right)^h - \left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h \tilde{\eta}\right)\right] \mathrm{d}t \\
& + \int_0^T \left(\tfrac{\partial U_\varepsilon}{\partial t}, (\pi^h - I)\,\eta\right) \mathrm{d}t \\
& + \int_0^T \left(\tfrac{\partial U_\varepsilon}{\partial t}, \eta\right) \mathrm{d}t \\
:= & \; I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}\,. \qquad (3.3.30)
\end{aligned}$$

Using (2.4.19), (3.1.10), (2.4.16), the Hölder's inequality and (3.3.8a) gives that

$$\begin{aligned}
|I_{1,1}| \equiv & \left| \int_0^T \left[\left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h[\eta - \tilde{\eta}]\right)^h - \left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h[\eta - \tilde{\eta}]\right)\right] \mathrm{d}t \right| \\
\leq & \int_0^T \left| \left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h[\eta - \tilde{\eta}]\right)^h - \left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h[\eta - \tilde{\eta}]\right) \right| \mathrm{d}t \\
\leq & \; C\,h \int_0^T \left\| \tfrac{\partial U_\varepsilon}{\partial t} \right\|_0 \left| \pi^h[\eta - \tilde{\eta}] \right|_1 \mathrm{d}t \\
\leq & \; C \int_0^T \left\| \mathcal{G} \tfrac{\partial U_\varepsilon}{\partial t} \right\|_1 \|\eta - \tilde{\eta}\|_1 \,\mathrm{d}t \\
\leq & \; C \left\| \mathcal{G} \tfrac{\partial U_\varepsilon}{\partial t} \right\|_{L^2(0,T;H^1(\Omega))} \|\eta - \tilde{\eta}\|_{L^2(0,T;H^1(\Omega))} \\
\leq & \; C \,\|\eta - \tilde{\eta}\|_{L^2(0,T;H^1(\Omega))}\,. \qquad (3.3.31)
\end{aligned}$$

It also follows from (2.4.19), (2.4.16), the Hölder's inequality and (3.3.8a) that

$$\begin{aligned}
|I_{1,2}| \equiv & \left| \int_0^T \left[\left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h \tilde{\eta}\right)^h - \left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h \tilde{\eta}\right) \right] \mathrm{d}t \right| \\
\leq & \left| \int_0^T \left[\left(U_\varepsilon, \tfrac{\partial(\pi^h \tilde{\eta})}{\partial t}\right)^h - \left(U_\varepsilon, \tfrac{\partial(\pi^h \tilde{\eta})}{\partial t}\right) \right] \mathrm{d}t \right| \\
& + \left| (U_\varepsilon(\cdot,T), \pi^h \tilde{\eta}(\cdot,T))^h - (U_\varepsilon(\cdot,T), \pi^h \tilde{\eta}(\cdot,T)) \right| \\
& + \left| (U_\varepsilon(\cdot,0), \pi^h \tilde{\eta}(\cdot,0))^h - (U_\varepsilon(\cdot,0), \pi^h \tilde{\eta}(\cdot,0)) \right| \\
\leq & \; C\,h \int_0^T \|U_\varepsilon\|_0 \, |\tfrac{\partial(\pi^h \tilde{\eta})}{\partial t}|_1 \,\mathrm{d}t + C\,h\, \|U_\varepsilon(\cdot,T)\|_0 \left|\pi^h \tilde{\eta}(\cdot,T)\right|_1 \\
& + C\,h\, \|U_\varepsilon(\cdot,0)\|_0 \left|\pi^h \tilde{\eta}(\cdot,0)\right|_1 \\
\leq & \; C\,h\, \|U_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \, \|\tilde{\eta}\|_{H^1(0,T;H^1(\Omega))} \\
\leq & \; C\,h\, \|\tilde{\eta}\|_{H^1(0,T;H^1(\Omega))}\,, \qquad (3.3.32)
\end{aligned}$$

where the third inequality was obtained from (2.4.16) and exploiting the continuous embedding (see Robinson [58] page 190):

$$W^{1,p}(0,T;X) \hookrightarrow C([0,T];X) \qquad\qquad 1 \leq p \leq \infty\,;$$

namely,

$$\sup_{t\in[0,T]} \|\zeta(t)\|_X \leq \|\zeta\|_{W^{1,p}(0,T;X)} \qquad \text{for} \ \ \zeta \in W^{1,p}(0,T;X)\,. \tag{3.3.33}$$

To treat the term $I_{1,3}$, we observe using (3.1.8), the Hölder's inequality and the fifth bound in (3.3.8a) that

$$\begin{aligned}
|I_{1,3}| &\equiv \left| \int_0^T \left(\tfrac{\partial U_\varepsilon}{\partial t}, (\pi^h - I)\,\eta \right)\,\mathrm{d}t \right| \\
&\leq \int_0^T \left| \left\langle \tfrac{\partial U_\varepsilon}{\partial t}, (\pi^h - I)\,\eta \right\rangle \right|\,\mathrm{d}t \\
&\leq \left\| \tfrac{\partial U_\varepsilon}{\partial t} \right\|_{L^2(0,T;(H^1(\Omega))')} \left\| (\pi^h - I)\,\eta \right\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\,\left\| (\pi^h - I)\,\eta \right\|_{L^2(0,T;H^1(\Omega))}\,.
\end{aligned} \tag{3.3.34}$$

From (3.1.8) and the weak convergence result (3.3.6c) we have, for all $\eta \in L^2(0,T;H^1(\Omega))$, that

$$I_{1,4} \equiv \int_0^T \left(\tfrac{\partial U_\varepsilon}{\partial t}, \eta \right)\,\mathrm{d}t = \int_0^T \left\langle \tfrac{\partial U_\varepsilon}{\partial t}, \eta \right\rangle\,\mathrm{d}t \quad \longrightarrow \int_0^T \left\langle \tfrac{\partial u}{\partial t}, \eta \right\rangle\,\mathrm{d}t \quad \text{as } h \to 0\,. \tag{3.3.35}$$

Combining (3.3.30)-(3.3.32), (3.3.34), (3.3.35), the denseness of $H^1(0,T;H^1(\Omega))$ in $L^2(0,T;H^1(\Omega))$ and (2.4.17) yields for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\int_0^T \left(\tfrac{\partial U_\varepsilon}{\partial t}, \pi^h\eta \right)^h\,\mathrm{d}t \quad \longrightarrow \int_0^T \left\langle \tfrac{\partial u}{\partial t}, \eta \right\rangle\,\mathrm{d}t \quad \text{as } h \to 0\,. \tag{3.3.36}$$

With the aid of the Hölder's inequality, (3.3.8a) and (2.4.17) we obtain for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned}
\left| \int_0^T (\,\nabla U_\varepsilon^+, \nabla(\pi^h - I)\,\eta)\,\mathrm{d}t \right| &\leq \int_0^T \left| U_\varepsilon^+ \right|_1 \left| (\pi^h - I)\,\eta \right|_1\,\mathrm{d}t \\
&\leq \left\| U_\varepsilon^+ \right\|_{L^2(0,T;H^1(\Omega))} \left\| (\pi^h - I)\,\eta \right\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\,\left\| (\pi^h - I)\,\eta \right\|_{L^2(0,T;H^1(\Omega))} \quad \longrightarrow 0 \quad \text{as } h \to 0\,.
\end{aligned} \tag{3.3.37}$$

Noting (3.3.37) and (3.3.6a) yields for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\int_0^T (\nabla U_\varepsilon^+, \nabla \pi^h \eta)\,\mathrm{d}t \;=\; \int_0^T (\nabla U_\varepsilon^+, \nabla(\pi^h - I)\,\eta)\,\mathrm{d}t \;+\int_0^T (\nabla U_\varepsilon^+, \nabla \eta)\,\mathrm{d}t$$
$$\longrightarrow \int_0^T (\nabla u, \nabla \eta)\,\mathrm{d}t \quad \text{as } h \to 0\,. \qquad (3.3.38)$$

We have for all $\eta \in L^2(0,T;H^1(\Omega))$ and for all $\tilde{\eta} \in L^\infty(0,T;W^{1,\infty}(\Omega))$ that

$$\begin{aligned}
&\int_0^T \left(\Lambda_\varepsilon(U_\varepsilon^+)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla \pi^h \eta\right)\,\mathrm{d}t\\
&\quad= \int_0^T \left(\Lambda_\varepsilon(U_\varepsilon^+)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla(\pi^h - I)\,\eta\right)\,\mathrm{d}t\\
&\qquad+ \int_0^T \left(\left[\Lambda_\varepsilon(U_\varepsilon^+) - \phi(u)\,\mathcal{I}\right]\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla(\eta - \tilde{\eta})\right)\,\mathrm{d}t\\
&\qquad+ \int_0^T \left(\left[\Lambda_\varepsilon(U_\varepsilon^+) - \phi(u)\,\mathcal{I}\right]\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla\tilde{\eta}\right)\,\mathrm{d}t\\
&\qquad+ \int_0^T \left(\phi(u)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla\eta\right)\,\mathrm{d}t\\
&\quad:= I_{2,1} + I_{2,2} + I_{2,3} + I_{2,4}.
\end{aligned} \qquad (3.3.39)$$

On noting the generalized Hölder's inequality and (3.3.8a)-(3.3.8b) we have

$$\begin{aligned}
|I_{2,1}| &\equiv \left|\int_0^T \left(\Lambda_\varepsilon(U_\varepsilon^+)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla(\pi^h - I)\,\eta\right)\,\mathrm{d}t\right|\\
&\le \left\|\Lambda_\varepsilon(U_\varepsilon^+)\right\|_{L^\infty(\Omega_T)} \left\|U_\varepsilon^+ + V_\varepsilon^+\right\|_{L^2(0,T;H^1(\Omega))} \left\|(\pi^h - I)\,\eta\right\|_{L^2(0,T;H^1(\Omega))}\\
&\le C\,\left\|(\pi^h - I)\,\eta\right\|_{L^2(0,T;H^1(\Omega))}\,.
\end{aligned} \qquad (3.3.40)$$

Similarly to the treatment of the term $I_{2,1}$, we have from the generalized Hölder's inequality, (3.3.8a)-(3.3.8b) and (2.2.1) that

$$\begin{aligned}
|I_{2,2}| &\equiv \left|\int_0^T \left(\left[\Lambda_\varepsilon(U_\varepsilon^+) - \phi(u)\,\mathcal{I}\right]\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla(\eta - \tilde{\eta})\right)\,\mathrm{d}t\right|\\
&\le \left\|\Lambda_\varepsilon(U_\varepsilon^+) - \phi(u)\,\mathcal{I}\right\|_{L^\infty(\Omega_T)} \left\|U_\varepsilon^+ + V_\varepsilon^+\right\|_{L^2(0,T;H^1(\Omega))} \|\eta - \tilde{\eta}\|_{L^2(0,T;H^1(\Omega))}\\
&\le C\,\|\eta - \tilde{\eta}\|_{L^2(0,T;H^1(\Omega))}\,.
\end{aligned} \qquad (3.3.41)$$

We also have that

$$\begin{aligned}
|I_{2,3}| &\equiv \left|\int_0^T \left(\left[\Lambda_\varepsilon(U_\varepsilon^+) - \phi(u)\,\mathcal{I}\right]\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla\tilde{\eta}\right)\,\mathrm{d}t\right|\\
&\le \left\|\Lambda_\varepsilon(U_\varepsilon^+) - \phi(u)\,\mathcal{I}\right\|_{L^2(\Omega_T)} \left\|U_\varepsilon^+ + V_\varepsilon^+\right\|_{L^2(0,T;H^1(\Omega))} \|\nabla\tilde{\eta}\|_{L^\infty(\Omega_T)}\\
&\le C\,\left\|\Lambda_\varepsilon(U_\varepsilon^+) - \phi(u)\,\mathcal{I}\right\|_{L^2(\Omega_T)} \|\tilde{\eta}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}\,.
\end{aligned} \qquad (3.3.42)$$

As the function $\phi(s)$ is bounded, we obtain from (3.3.6a) and (3.3.7a) for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$I_{2,4} \equiv \int_0^T \big(\phi(u)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla\eta\big)\;\mathrm{d}t \;\to \int_0^T \left(\phi(u)\,\nabla(u+v), \nabla\eta\right)\,\mathrm{d}t \quad \text{as } h \to 0\,. \tag{3.3.43}$$

Combining (3.3.39)-(3.3.43), (2.4.17), the denseness of the space $L^\infty(0,T;W^{1,\infty}(\Omega))$ in $L^2(0,T;H^1(\Omega))$ and (3.3.6g) yields for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\int_0^T \big(\Lambda_\varepsilon(U_\varepsilon^+)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla\pi^h\eta\big)\;\mathrm{d}t \;\to \int_0^T \left(\phi(u)\,\nabla(u+v), \nabla\eta\right)\,\mathrm{d}t \quad \text{as } h \to 0\,. \tag{3.3.44}$$

It remains to show the convergence of the reaction term in (3.3.29). We have from (2.4.19), the Hölder's inequality, (2.4.16) and (3.3.8a) for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned}
&\left|\int_0^T \Big[\big(U_\varepsilon^+, \pi^h\eta\big)^h - \big(U_\varepsilon^+, \pi^h\eta\big)\Big]\;\mathrm{d}t\right| + \left|\int_0^T \big(U_\varepsilon^+, (\pi^h - I)\,\eta\big)\;\mathrm{d}t\right| \\
&\qquad \le C\,h \int_0^T \left\|U_\varepsilon^+\right\|_0 \left|\pi^h\eta\right|_1\,\mathrm{d}t + \int_0^T \left\|U_\varepsilon^+\right\|_0 \left\|(\pi^h - I)\,\eta\right\|_0\,\mathrm{d}t \\
&\qquad \le C\,h \int_0^T \left\|U_\varepsilon^+\right\|_0 |\eta|_1\,\mathrm{d}t \\
&\qquad \le C\,h \left\|U_\varepsilon^+\right\|_{L^2(\Omega_T)} \|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\qquad \le C\,h \,\|\eta\|_{L^2(0,T;H^1(\Omega))} \;\longrightarrow 0 \quad \text{as } h \to 0\,.
\end{aligned} \tag{3.3.45}$$

Combining (3.3.45) and (3.3.6a) yields for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned}
\int_0^T \big(U_\varepsilon^+, \pi^h\eta\big)^h\;\mathrm{d}t &= \int_0^T \Big[\big(U_\varepsilon^+, \pi^h\eta\big)^h - \big(U_\varepsilon^+, \pi^h\eta\big)\Big]\;\mathrm{d}t \\
&\quad + \int_0^T \big(U_\varepsilon^+, (\pi^h - I)\,\eta\big)\;\mathrm{d}t \;+ \int_0^T \big(U_\varepsilon^+, \eta\big)\;\mathrm{d}t \;\to \int_0^T (u, \eta)\;\mathrm{d}t \quad \text{as } h \to 0\,.
\end{aligned} \tag{3.3.46}$$

With the aid of Lemma 3.3.2, we now consider the convergence of the non-linear reaction terms in (3.3.29). First, note from (2.4.3) for all $\eta \in L^2(0,T;H^1(\Omega))$ and for all $\tilde{\eta} \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\begin{aligned}
\int_0^T \left(\phi_\varepsilon(U_\varepsilon^+)\,\psi_\varepsilon(V_\varepsilon^-),\pi^h\eta\right)^h \,\mathrm{d}t &= \int_0^T \left(\pi^h\phi_\varepsilon(U_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),\pi^h\eta\right)^h \,\mathrm{d}t \\
&= \int_0^T \Big[\left(\pi^h\phi_\varepsilon(U_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),\pi^h\eta\right)^h - \left(\pi^h\phi_\varepsilon(U_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),\pi^h\eta\right)\Big] \,\mathrm{d}t \\
&\quad + \int_0^T \left(\pi^h\phi_\varepsilon(U_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-) - \phi(u)\,v,\pi^h[\eta-\tilde{\eta}]\right) \mathrm{d}t \\
&\quad + \int_0^T \left(\pi^h\phi_\varepsilon(U_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-) - \phi(u)\,v,\pi^h\tilde{\eta}\right) \mathrm{d}t \\
&\quad + \int_0^T \left(\phi(u)\,v,(\pi^h - I)\,\eta\right) \mathrm{d}t \\
&\quad + \int_0^T (\phi(u)\,v,\eta) \,\mathrm{d}t \\
&:= I_{3,1} + I_{3,2} + I_{3,3} + I_{3,4} + I_{3,5}\,.
\end{aligned} \tag{3.3.47}$$

It follows from (2.4.20), (2.4.14), (2.4.6), the Hölder's inequality, (2.4.16), (3.3.8a)-(3.3.8b) and the embedding $L^\beta(\Omega_T) \hookrightarrow L^2(\Omega_T)$ that

$$\begin{aligned}
|I_{3,1}| &\equiv \left|\int_0^T \Big[\left(\pi^h\phi_\varepsilon(U_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),\pi^h\eta\right)^h - \left(\pi^h\phi_\varepsilon(U_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),\pi^h\eta\right)\Big] \,\mathrm{d}t\right| \\
&\leq C\,h^2 \int_0^T \left\|\pi^h\phi_\varepsilon(U_\varepsilon^+)\right\|_{1,\infty} \left\|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right\|_1 \left\|\pi^h\eta\right\|_1 \,\mathrm{d}t \\
&\leq C\,h \int_0^T \left\|\pi^h\phi_\varepsilon(U_\varepsilon^+)\right\|_{0,\infty} \left[\left\|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right\|_0 + \left|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right|_1\right] \left\|\pi^h\eta\right\|_1 \,\mathrm{d}t \\
&\leq C\,h \left[\left\|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right\|_{L^2(\Omega_T)} + \left\|V_\varepsilon^-\right\|_{L^2(0,T;H^1(\Omega))}\right] \left\|\pi^h\eta\right\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\,h\,\|\eta\|_{L^2(0,T;H^1(\Omega))} \longrightarrow 0 \quad \text{as } h \to 0\,.
\end{aligned} \tag{3.3.48}$$

Using the Hölder's inequality, (2.4.16), (2.2.1), (3.3.8a)-(3.3.8b) and (3.3.5b) gives that

$$\begin{aligned}
|I_{3,2}| &\equiv \left|\int_0^T \left(\pi^h\phi_\varepsilon(U_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-) - \phi(u)\,v,\pi^h[\eta-\tilde{\eta}]\right) \mathrm{d}t\right| \\
&\leq \left\|\pi^h\phi_\varepsilon(U_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-) - \phi(u)\,v\right\|_{L^2(\Omega_T)} \left\|\pi^h[\eta-\tilde{\eta}]\right\|_{L^2(\Omega_T)} \\
&\leq C\,\left(\left\|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right\|_{L^2(\Omega_T)} + \|v\|_{L^2(\Omega_T)}\right) \|\eta-\tilde{\eta}\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\,\|\eta-\tilde{\eta}\|_{L^2(0,T;H^1(\Omega))}\,.
\end{aligned} \tag{3.3.49}$$

We also use the Hölder's inequality and (2.4.16) to obtain that

$$
\begin{aligned}
|I_{3,3}| &\equiv \left| \int_0^T \left(\pi^h \phi_\varepsilon(U_\varepsilon^+)\, \pi^h \psi_\varepsilon(V_\varepsilon^-) - \phi(u)\, v, \pi^h \tilde{\eta} \right) \mathrm{d}t \right| \\
&\leq \left\| \pi^h \phi_\varepsilon(U_\varepsilon^+)\, \pi^h \psi_\varepsilon(V_\varepsilon^-) - \phi(u)\, v \right\|_{L^q(\Omega_T)} \left\| \pi^h \tilde{\eta} \right\|_{L^{q'}(\Omega_T)} \\
&\leq C \left\| \pi^h \phi_\varepsilon(U_\varepsilon^+)\, \pi^h \psi_\varepsilon(V_\varepsilon^-) - \phi(u)\, v \right\|_{L^q(\Omega_T)} \|\tilde{\eta}\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} .
\end{aligned} \tag{3.3.50}
$$

Again, on noting (2.2.1) and (3.3.5b), we use (2.4.16) to deduce that

$$
\begin{aligned}
|I_{3,4}| &\equiv \left| \int_0^T \left(\phi(u)\, v, (\pi^h - I)\, \eta \right) \mathrm{d}t \right| \\
&\leq M \int_0^T \|v\|_0 \left\| (\pi^h - I)\, \eta \right\|_0 \mathrm{d}t \\
&\leq C\, h\, \|v\|_{L^2(\Omega_T)} \|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\, h\, \|\eta\|_{L^2(0,T;H^1(\Omega))} \longrightarrow 0 \quad \text{as } h \to 0 .
\end{aligned} \tag{3.3.51}
$$

From (3.3.47)-(3.3.51), the denseness of $L^{q'}(0,T;W^{1,q'}(\Omega))$ in $L^2(0,T;H^1(\Omega))$ and (3.3.24b) we have for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$
\int_0^T \left(\phi_\varepsilon(U_\varepsilon^+)\, \psi_\varepsilon(V_\varepsilon^-), \pi^h \eta \right)^h \mathrm{d}t \longrightarrow I_{3,5} \equiv \int_0^T \left(\phi(u)\, v, \eta \right) \mathrm{d}t \quad \text{as } h \to 0 . \tag{3.3.52}
$$

By considering the convergence result (3.3.24a), one can adapt the same argument used for deriving (3.3.52) in order to show, for all $\eta \in L^2(0,T;H^1(\Omega))$, that

$$
\int_0^T \left(U_\varepsilon^+\, \phi_\varepsilon(U_\varepsilon^-), \pi^h \eta \right)^h \mathrm{d}t \longrightarrow \int_0^T \left(u\, \phi(u), \eta \right) \mathrm{d}t \quad \text{as } h \to 0 . \tag{3.3.53}
$$

Therefore, the desired result (3.3.28a) follows by combining (3.3.29), (3.3.36), (3.3.38), (3.3.44), (3.3.46), (3.3.52) and (3.3.53).

Similarly to the proof of (3.3.28a), we now show briefly that the solution $\{u, v\}$ satisfies (3.3.28b). For any $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$, we set $\chi \equiv \pi^h \eta$ as a test function in (3.3.4a) yielding

$$
\begin{aligned}
&\int_0^T \left[\left(\tfrac{\partial V_\varepsilon}{\partial t}, \pi^h \eta \right)^h + D \left(\nabla V_\varepsilon^+, \nabla \pi^h \eta \right) + \left(\Xi_\varepsilon(V_\varepsilon^+)\, \nabla (U_\varepsilon^+ + V_\varepsilon^+), \nabla \pi^h \eta \right) \right] \mathrm{d}t \\
&\qquad = \int_0^T \left(\gamma\, V_\varepsilon^+ - \psi_\varepsilon(V_\varepsilon^+) \left[\phi_\varepsilon(U_\varepsilon^-) + \psi_\varepsilon(V_\varepsilon^-) \right], \pi^h \eta \right)^h \mathrm{d}t .
\end{aligned} \tag{3.3.54}
$$

For all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ and for all $\tilde{\eta} \in W^{1,q'}(0,T;W^{1,\infty}(\Omega))$ we have that

$$\begin{aligned}
\int_0^T \left(\tfrac{\partial V_\varepsilon}{\partial t}, \pi^h \eta\right)^h \, \mathrm{d}t &= \int_0^T \left[\left(\tfrac{\partial V_\varepsilon}{\partial t}, \pi^h[\eta - \tilde{\eta}]\right)^h - \left(\tfrac{\partial V_\varepsilon}{\partial t}, \pi^h[\eta - \tilde{\eta}]\right) \right] \mathrm{d}t \\
&\quad + \int_0^T \left[\left(\tfrac{\partial V_\varepsilon}{\partial t}, \pi^h \tilde{\eta}\right)^h - \left(\tfrac{\partial V_\varepsilon}{\partial t}, \pi^h \tilde{\eta}\right) \right] \mathrm{d}t \\
&\quad + \int_0^T \left(\tfrac{\partial V_\varepsilon}{\partial t}, (\pi^h - I)\,\eta\right) \mathrm{d}t \\
&\quad + \int_0^T \left(\tfrac{\partial V_\varepsilon}{\partial t}, \eta\right) \mathrm{d}t \\
&:= I_{4,1} + I_{4,2} + I_{4,3} + I_{4,4} \,.
\end{aligned} \tag{3.3.55}$$

It follows from (2.4.19),(3.1.10), (2.4.16), the Hölder's inequality and the sixth bound in (3.3.8b) that

$$\begin{aligned}
|I_{4,1}| &\equiv \left| \int_0^T \left[\left(\tfrac{\partial V_\varepsilon}{\partial t}, \pi^h[\eta - \tilde{\eta}]\right)^h - \left(\tfrac{\partial V_\varepsilon}{\partial t}, \pi^h[\eta - \tilde{\eta}]\right) \right] \mathrm{d}t \right| \\
&\leq C\,h \int_0^T \left\| \tfrac{\partial V_\varepsilon}{\partial t} \right\|_{0,q} \left| \pi^h[\eta - \tilde{\eta}] \right|_{1,q'} \mathrm{d}t \\
&\leq C \int_0^T \left\| \mathcal{G}_q \tfrac{\partial V_\varepsilon}{\partial t} \right\|_{1,q} \|\eta - \tilde{\eta}\|_{1,q'} \, \mathrm{d}t \\
&\leq C \left\| \mathcal{G}_q \tfrac{\partial V_\varepsilon}{\partial t} \right\|_{L^q(0,T;W^{1,q}(\Omega))} \|\eta - \tilde{\eta}\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \\
&\leq C \, \|\eta - \tilde{\eta}\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \,.
\end{aligned} \tag{3.3.56}$$

From (2.4.19), (2.4.16), (3.3.33), the continuous embedding of $L^{q'}(0,T;W^{1,\infty}(\Omega))$ into $L^1(0,T;W^{1,\infty}(\Omega))$ and (3.3.8b) we obtain that

$$\begin{aligned}
|I_{4,2}| &\equiv \left| \int_0^T \left[\left(\tfrac{\partial V_\varepsilon}{\partial t}, \pi^h \tilde{\eta}\right)^h - \left(\tfrac{\partial V_\varepsilon}{\partial t}, \pi^h \tilde{\eta}\right) \right] \mathrm{d}t \right| \\
&\leq C\,h \int_0^T \|V_\varepsilon\|_{0,1} \left| \tfrac{\partial(\pi^h \tilde{\eta})}{\partial t} \right|_{1,\infty} \mathrm{d}t + C\,h\, \|V_\varepsilon(\cdot,T)\|_{0,1} \left| \pi^h \tilde{\eta}(\cdot,T) \right|_{1,\infty} \\
&\qquad + C\,h\, \|V_\varepsilon(\cdot,0)\|_{0,1} \left| \pi^h \tilde{\eta}(\cdot,0) \right|_{1,\infty} \\
&\leq C\,h\, \|V_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \, \|\tilde{\eta}\|_{W^{1,q'}(0,T;W^{1,\infty}(\Omega))} \\
&\leq C\,h\, \|\tilde{\eta}\|_{W^{1,q'}(0,T;W^{1,\infty}(\Omega))} \longrightarrow 0 \quad \text{as } h \to 0 \,.
\end{aligned} \tag{3.3.57}$$

On noting (3.1.5), the Hölder's inequality and the fifth bound in (3.3.8b) we have that

$$
\begin{aligned}
|I_{4,3}| \equiv \left| \int_0^T \left(\tfrac{\partial V_\varepsilon}{\partial t}, (\pi^h - I)\,\eta \right) \mathrm{d}t \right| &\le \int_0^T \left| \left\langle \tfrac{\partial V_\varepsilon}{\partial t}, (\pi^h - I)\,\eta \right\rangle_{q'} \right| \mathrm{d}t \\
&\le \left\| \tfrac{\partial V_\varepsilon}{\partial t} \right\|_{L^q(0,T;(W^{1,q'}(\Omega))')} \left\| (\pi^h - I)\,\eta \right\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \\
&\le C \left\| (\pi^h - I)\,\eta \right\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}. \qquad (3.3.58)
\end{aligned}
$$

Noting (3.3.55)-(3.3.58), the denseness of $W^{1,q'}(0,T;W^{1,\infty}(\Omega))$ in $L^{q'}(0,T;W^{1,q'}(\Omega))$, (2.4.17) and the convergence in (3.3.7b) yields for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$
\int_0^T \left(\tfrac{\partial V_\varepsilon}{\partial t}, \pi^h \eta \right)^h \mathrm{d}t \quad \longrightarrow \int_0^T \left\langle \tfrac{\partial v}{\partial t}, \eta \right\rangle_{q'} \mathrm{d}t \quad \text{as } h \to 0\,. \qquad (3.3.59)
$$

It follows from the Hölder's inequality, the continuous embedding $L^2(0,T;H^1(\Omega)) \hookrightarrow L^q(0,T;W^{1,q}(\Omega))$, (3.3.8b) and (2.4.17) for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$
\begin{aligned}
\left| \int_0^T (\nabla V_\varepsilon^+, \nabla(\pi^h - I)\,\eta)\,\mathrm{d}t \right| &\le \int_0^T \left| V_\varepsilon^+ \right|_{1,q} \left| (\pi^h - I)\,\eta \right|_{1,q'} \mathrm{d}t \\
&\le \left\| V_\varepsilon^+ \right\|_{L^q(0,T;W^{1,q}(\Omega))} \left\| (\pi^h - I)\,\eta \right\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \\
&\le C \left\| (\pi^h - I)\,\eta \right\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \quad \longrightarrow 0 \quad \text{as } h \to 0\,.
\end{aligned}
\qquad (3.3.60)
$$

Thereby, we obtain from (3.3.60) and (3.3.7a) for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$
\begin{aligned}
\int_0^T (\nabla V_\varepsilon^+, \nabla \pi^h \eta)\,\mathrm{d}t &= \int_0^T (\nabla V_\varepsilon^+, \nabla(\pi^h - I)\,\eta)\,\mathrm{d}t + \int_0^T (\nabla V_\varepsilon^+, \nabla \eta)\,\mathrm{d}t \\
&\longrightarrow \int_0^T (\nabla v, \nabla \eta)\,\mathrm{d}t \quad \text{as } h \to 0\,. \qquad (3.3.61)
\end{aligned}
$$

Similarly to (3.3.39), we have for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ and for all $\tilde{\eta} \in L^\infty(0,T;W^{1,q'}(\Omega))$ that

$$
\begin{aligned}
&\int_0^T \left(\Xi_\varepsilon(V_\varepsilon^+)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla \pi^h \eta \right) \mathrm{d}t \\
&\quad = \int_0^T \left(\Xi_\varepsilon(V_\varepsilon^+)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla(\pi^h - I)\,\eta \right) \mathrm{d}t \\
&\qquad + \int_0^T \left(\left[\Xi_\varepsilon(V_\varepsilon^+) - v\,\mathcal{I} \right] \nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla(\eta - \tilde{\eta}) \right) \mathrm{d}t \\
&\qquad + \int_0^T \left(\left[\Xi_\varepsilon(V_\varepsilon^+) - v\,\mathcal{I} \right] \nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla \tilde{\eta} \right) \mathrm{d}t \\
&\qquad + \int_0^T \left(v\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla \eta \right) \mathrm{d}t \\
&\quad := I_{5,1} + I_{5,2} + I_{5,3} + I_{5,4}. \qquad (3.3.62)
\end{aligned}
$$

It follows from the generalized Hölder's inequality, (3.3.8a)-(3.3.8b), (3.3.5b) and (2.4.17) for $\beta = \frac{2\,(d+1)}{d}$ that

$$
\begin{aligned}
|I_{5,1}| &\equiv \left| \int_0^T \big(\Xi_\varepsilon(V_\varepsilon^+)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla(\pi^h - I)\,\eta\big)\ \mathrm{d}t \right| \\
&\leq \left\|\Xi_\varepsilon(V_\varepsilon^+)\right\|_{L^\beta(\Omega_T)} \left\|U_\varepsilon^+ + V_\varepsilon^+\right\|_{L^2(0,T;H^1(\Omega))} \left\|(\pi^h - I)\,\eta\right\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \\
&\leq C\, \left\|(\pi^h - I)\,\eta\right\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \quad \longrightarrow 0 \quad \text{as } h \to 0\,,
\end{aligned} \tag{3.3.63}
$$

and

$$
\begin{aligned}
|I_{5,2}| &\equiv \left| \int_0^T \big(\big[\Xi_\varepsilon(V_\varepsilon^+) - v\,\mathcal{I}\big]\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla(\eta - \tilde{\eta})\big)\ \mathrm{d}t \right| \\
&\leq \left\|\Xi_\varepsilon(V_\varepsilon^+) - v\,\mathcal{I}\right\|_{L^\beta(\Omega_T)} \left\|U_\varepsilon^+ + V_\varepsilon^+\right\|_{L^2(0,T;H^1(\Omega))} \|\eta - \tilde{\eta}\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \\
&\leq C\, \|\eta - \tilde{\eta}\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}\,.
\end{aligned} \tag{3.3.64}
$$

Once again, using the generalized Hölder's inequality and (3.3.8a)-(3.3.8b), we have that

$$
\begin{aligned}
|I_{5,3}| &\equiv \left| \int_0^T \big(\big[\Xi_\varepsilon(V_\varepsilon^+) - v\,\mathcal{I}\big]\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla\tilde{\eta}\big)\ \mathrm{d}t \right| \\
&\leq \left\|\Xi_\varepsilon(V_\varepsilon^+) - v\,\mathcal{I}\right\|_{L^2(0,T;L^\beta(\Omega)} \left\|U_\varepsilon^+ + V_\varepsilon^+\right\|_{L^2(0,T;H^1(\Omega))} \|\tilde{\eta}\|_{L^\infty(0,T;W^{1,q'}(\Omega))} \\
&\leq C\, \left\|\Xi_\varepsilon(V_\varepsilon^+) - v\,\mathcal{I}\right\|_{L^2(0,T;L^\beta(\Omega)} \|\tilde{\eta}\|_{L^\infty(0,T;W^{1,q'}(\Omega))}\,.
\end{aligned} \tag{3.3.65}
$$

Combining (3.3.62)-(3.3.65), the denseness of $L^\infty(0,T;W^{1,q'}(\Omega))$ in $L^{q'}(0,T;W^{1,q'}(\Omega))$, (3.3.7f) and (3.3.7a) yields, after noting that $v \in L^\beta(\Omega_T)$, for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$
\int_0^T \big(\Xi_\varepsilon(V_\varepsilon^+)\,\nabla(U_\varepsilon^+ + V_\varepsilon^+), \nabla\pi^h\eta\big)\ \mathrm{d}t \ \to \int_0^T (v\,\nabla(u+v), \nabla\eta)\ \mathrm{d}t \quad \text{as } h \to 0\,. \tag{3.3.66}
$$

It remains to show the convergence of the reaction term in (3.3.54). Similarly to (3.3.45)-(3.3.46), we obtain from (2.4.19), the Hölder's inequality, (2.4.16), (3.3.8b) and (3.3.7a) for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$
\begin{aligned}
\int_0^T \big(V_\varepsilon^+, \pi^h\eta\big)^h\ \mathrm{d}t &= \int_0^T \Big[\big(V_\varepsilon^+, \pi^h\eta\big)^h - \big(V_\varepsilon^+, \pi^h\eta\big)\Big]\ \mathrm{d}t \\
&\quad + \int_0^T \big(V_\varepsilon^+, (\pi^h - I)\,\eta\big)\ \mathrm{d}t\ + \int_0^T \big(V_\varepsilon^+, \eta\big)\ \mathrm{d}t\ \to \int_0^T (v, \eta)\ \mathrm{d}t \quad \text{as } h \to 0\,.
\end{aligned} \tag{3.3.67}
$$

It follows from (2.4.20), (2.4.14), (2.4.16), (2.4.6), the generalized Hölder's inequality and (3.3.8b) for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\begin{aligned}
&\left|\int_0^T \left[\left(\pi^h\psi_\varepsilon(V_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),\pi^h\eta\right)^h - \left(\pi^h\psi_\varepsilon(V_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),\pi^h\eta\right)\right]\,\mathrm{d}t\right| \\
&\qquad\qquad + \left|\int_0^T \left(\pi^h\psi_\varepsilon(V_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),(\pi^h - I)\,\eta\right)\,\mathrm{d}t\right| \\
&\leq C\,h^2 \int_0^T \left\|\pi^h\psi_\varepsilon(V_\varepsilon^+)\right\|_{1,\beta}\left\|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right\|_1\left\|\pi^h\eta\right\|_{1,q'}\,\mathrm{d}t \\
&\qquad\qquad + \int_0^T \left\|\pi^h\psi_\varepsilon(V_\varepsilon^+)\right\|_{0,\beta}\left\|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right\|_0\left\|(\pi^h - I)\,\eta\right\|_{0,q'}\,\mathrm{d}t \\
&\leq C\,h \int_0^T \left\|\pi^h\psi_\varepsilon(V_\varepsilon^+)\right\|_{0,\beta}\left[\left\|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right\|_0 + \left|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right|_1\right]\|\eta\|_{1,q'}\,\mathrm{d}t \\
&\qquad\qquad + C\,h\int_0^T \left\|\pi^h\psi_\varepsilon(V_\varepsilon^+)\right\|_{0,\beta}\left\|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right\|_0\|\eta\|_{1,q'}\,\mathrm{d}t \\
&\leq C\,h \int_0^T \left\|\pi^h\psi_\varepsilon(V_\varepsilon^+)\right\|_{0,\beta}\left[\left\|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right\|_0 + \left|V_\varepsilon^-\right|_1\right]\|\eta\|_{1,q'}\,\mathrm{d}t \\
&\leq C\,h\,\left\|\pi^h\psi_\varepsilon(V_\varepsilon^+)\right\|_{L^\beta(\Omega_T)}\left[\left\|\pi^h\psi_\varepsilon(V_\varepsilon^-)\right\|_{L^2(\Omega_T)} + \left\|V_\varepsilon^-\right\|_{L^2(0,T;H^1(\Omega))}\right]\|\eta\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \\
&\leq C\,h\,\|\eta\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}\,. \qquad\qquad (3.3.68)
\end{aligned}$$

Noting (2.4.3), (3.3.68) and (3.3.24c) yields for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\begin{aligned}
&\int_0^T \left(\pi^h\psi_\varepsilon(V_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),\pi^h\eta\right)^h\,\mathrm{d}t \\
&\qquad = \int_0^T \left[\left(\pi^h\psi_\varepsilon(V_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),\pi^h\eta\right)^h - \left(\pi^h\psi_\varepsilon(V_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),\pi^h\eta\right)\right]\,\mathrm{d}t \\
&\qquad + \int_0^T \left(\pi^h\psi_\varepsilon(V_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),(\pi^h - I)\,\eta\right)\,\mathrm{d}t + \int_0^T \left(\pi^h\psi_\varepsilon(V_\varepsilon^+)\,\pi^h\psi_\varepsilon(V_\varepsilon^-),\eta\right)\,\mathrm{d}t \\
&\qquad\qquad \longrightarrow \int_0^T \left(v^2,\eta\right)\,\mathrm{d}t \quad \text{as} \quad h \to 0\,. \qquad\qquad (3.3.69)
\end{aligned}$$

Similarly to (3.3.68) and (3.3.69), we can easily show using (3.3.24b) for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\int_0^T \left(\pi^h\psi_\varepsilon(V_\varepsilon^+)\,\pi^h\phi_\varepsilon(U_\varepsilon^-),\pi^h\eta\right)^h\,\mathrm{d}t \longrightarrow \int_0^T (v\,\phi(u),\eta)\,\mathrm{d}t \quad \text{as} \quad h \to 0\,. \qquad (3.3.70)$$

Hence, combining (3.3.54), (3.3.59), (3.3.61), (3.3.66), (3.3.67), (3.3.69) and (3.3.70) leads to the desired result (3.3.28b).

This completes the proof of the main theorem in this chapter. □

Remark 3.3.1 We note from (2.3.10b), (2.3.11) and (2.3.15) that for all $\varepsilon \in (0, e^{-1})$:

$$s\, G'_\varepsilon(s) \geq \psi_\varepsilon(s)\, G'_\varepsilon(s) \geq s - 1 \qquad \text{for all } s \in \mathbb{R}\,.$$

Taking this into account, the results in Theorem 3.3.3 can be achieved with the reaction term $g_{M,\varepsilon}(u_\varepsilon, v_\varepsilon)$, in ($\mathrm{P}_{M,\varepsilon}$), and the term $\gamma\, V^n_\varepsilon - \psi_\varepsilon(V^n_\varepsilon)\ [\phi_\varepsilon(U^{n-1}_\varepsilon) + \psi_\varepsilon(V^{n-1}_\varepsilon)]$, in ($\mathrm{P}^{h,\Delta t}_{M,\varepsilon}$), replaced by

$$\gamma\, v_\varepsilon - v_\varepsilon\ [\phi_\varepsilon(u_\varepsilon) + \psi_\varepsilon(v_\varepsilon)] \quad \text{and} \quad \gamma\, V^n_\varepsilon - V^n_\varepsilon\ \left[\phi_\varepsilon(U^{n-1}_\varepsilon) + \psi_\varepsilon(V^{n-1}_\varepsilon)\right]\ ,$$

respectively.

Remark 3.3.2 We have from (2.3.4b), (2.3.5), (2.3.11) and (2.3.9), for sufficiently small ε, that

$$\psi_\varepsilon(s)\, F'_\varepsilon(s) \geq \phi_\varepsilon(s)\, F'_\varepsilon(s) \geq s - 1 \qquad \text{for all } s \in \mathbb{R}\,.$$

Noting this, the results in Theorem 3.3.3 can be also achieved with the reaction term $f_{M,\varepsilon}(u_\varepsilon, v_\varepsilon)$, in ($\mathrm{P}_{M,\varepsilon}$), and the term $U^n_\varepsilon - U^n_\varepsilon\, \phi_\varepsilon(U^{n-1}_\varepsilon) - \phi_\varepsilon(U^n_\varepsilon)\, \psi_\varepsilon(V^{n-1}_\varepsilon)$, in ($\mathrm{P}^{h,\Delta t}_{M,\varepsilon}$), replaced by

$$u_\varepsilon - \phi_\varepsilon(u_\varepsilon)\ [\psi_\varepsilon(u_\varepsilon) + \psi_\varepsilon(v_\varepsilon)] \quad \text{and} \quad U^n_\varepsilon - \psi_\varepsilon(U^n_\varepsilon)\, \phi_\varepsilon(U^{n-1}_\varepsilon) - \phi_\varepsilon(U^n_\varepsilon)\, \psi_\varepsilon(V^{n-1}_\varepsilon)\,,$$

respectively. Where in this case we need to note, in addition to the convergence results stated in Theorem 3.3.3, that

$$\pi^h \psi_\varepsilon(U^\pm_\varepsilon)\ \to u \quad \text{in} \quad L^2(0,T; L^s(\Omega))\,.$$

Remark 3.3.3 In the problem (P_M) we assumed that $M \geq e$, however we can consider it for $M \in [1, e)$. For such choice of M, the property (2.3.8) does not hold and needs to be replaced by

$$s\, F'_\varepsilon(s) \leq \tfrac{5}{2}\, F_\varepsilon(s) + 1 \quad \forall s \in \mathbb{R} \qquad \text{holds for all } M \geq 1\,. \tag{3.3.71}$$

The proof of (3.3.71) for $s \leq M$ is exactly the same as the proof of (2.3.8). To conclude (3.3.71) for all $s \geq M$, define

$$J_\varepsilon(s) = \tfrac{5}{2}\, F_\varepsilon(s) - s\, F'_\varepsilon(s) + 1$$

with

$$J'_\varepsilon(s) = \tfrac{3}{2}\, F'_\varepsilon(s) - s\, F''_\varepsilon(s) \qquad \text{and} \qquad J'_\varepsilon(s) = 0 \Leftrightarrow s = 3\, M\, (1 - \ln M)\,.$$

For $M \in [1, e^{\frac{2}{3}}]$, we have $3\, M\, (1 - \ln M) \geq M$ and for all $s \geq M$

$$J_\varepsilon(s) \geq J_\varepsilon(\, 3\, M\, (1 - \ln M)\,) = -\tfrac{9\, M}{4}\, (1 - \ln M)^2 - \tfrac{5\, M}{4} + \tfrac{7}{2} \geq 0\,.$$

For $M \in [e^{\frac{2}{3}}, e)$, we have $3\, M\, (1 - \ln M) \leq M$ and for all $s \geq M$

$$J_\varepsilon(s) \geq J_\varepsilon(M) = \tfrac{3}{2}\, M\, (\ln M - 1) - M + \tfrac{7}{2} \geq 0\,.$$

As a result of the replacement of (2.3.8) by (3.3.71), the existence condition on the time discretization, for the n-th step of $(\mathrm{P}^{h,\Delta t}_{M,\varepsilon})$, in Theorem 2.4.7 for the case $M \in [1, e)$ will be

$$\Delta t_n \leq \tfrac{1}{\max\{9/2\,,\, 2\,\gamma+2\}}\,.$$

Therefore, the restriction considered for deriving the stability bounds on the approximations, in Lemma 3.2.2, will be replaced by

$$\Delta t \leq \tfrac{1-\delta}{\max\{9/2\,,\, 2\,\gamma+2\}}\,, \qquad \text{for some } \delta \in (0, 1)\,,$$

which is more severe than the restriction that required when $M \geq e$. In the experiments, in Chapter 5, we spend some time discussing the influence of the choices of the number M.

Chapter 4

The population model: Improved results

In this chapter we attempt to obtain more regular solutions of the problem (P) than the solutions derived by analysing the truncated problem (P_M) in Theorem 3.3.3. Based on the analysis in the previous chapters, the idea is to introduce and analyse an alternative problem to (P_M) which will be equivalent to the problem (P) in some sense. We do that briefly in three short sections with an emphasis only on the details that leads to further improvements on the solutions. In Section 4.1 a "fully" truncated alternative problem to (P) is presented. A corresponding regularized problem and entropy inequality is discussed. In Section 4.2 a practical fully discrete finite element approximation is proposed. The existence theorem of the approximate solutions is stated and a discrete analogue entropy inequality is derived. In Section 4.3 existence of a global weak solution to the "fully" truncated problem is established via studying the convergence of the approximate problem. Finally, in Section 4.4, in the absence of the reaction terms further features of the "fully" truncated problem are investigated. In particular, an $L^2(\Omega_T)$ estimate between the weak solution and the mean integral of the initial data has been obtained.

4.1 A "fully" truncated alternative problem

On noting the analysis of the problem (P_M), one expects to obtain more regularity by considering the following alternative problem to (P):

$(\widetilde{\mathbf{P}}_M)$ For fixed $M \geq e$, find $\{\tilde{u}(x,t), \tilde{v}(x,t)\} \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ such that

$$\frac{\partial \tilde{u}}{\partial t} = \nabla \cdot (\, D\, \nabla \tilde{u} + \phi(\tilde{u})\, \nabla(\tilde{u} + \tilde{v})) + \tilde{f}_M(\tilde{u}, \tilde{v}) \qquad \text{in } \Omega_T, \tag{4.1.1a}$$

$$\frac{\partial \tilde{v}}{\partial t} = \nabla \cdot (\, D\, \nabla \tilde{v} + \phi(\tilde{v})\, \nabla(\tilde{u} + \tilde{v})) + \tilde{g}_M(\tilde{u}, \tilde{v}) \qquad \text{in } \Omega_T, \tag{4.1.1b}$$

with boundary conditions

$$\left.\begin{aligned} &[\, D\, \nabla \tilde{u} + \phi(\tilde{u})\, \nabla(\tilde{u} + \tilde{v})] \cdot \nu = 0 \\ &[\, D\, \nabla \tilde{v} + \phi(\tilde{v})\, \nabla(\tilde{u} + \tilde{v})] \cdot \nu = 0 \end{aligned}\right. \qquad \text{on } \partial\Omega \times (0,T), \tag{4.1.1c}$$

and initial conditions

$$\tilde{u}(x,0) = u^0(x)\,, \quad \tilde{v}(x,0) = v^0(x) \qquad \forall\, x \in \Omega, \tag{4.1.1d}$$

where $\phi(s) := \phi_M(s)$ is defined by (2.2.1) and

$$\tilde{f}_M(\tilde{u}, \tilde{v}) := \tilde{u} - \tilde{u}\,(\, \phi(\tilde{u}) + \phi(\tilde{v})\,), \tag{4.1.1e}$$

$$\tilde{g}_M(\tilde{u}, \tilde{v}) := \gamma\, \tilde{v} - \tilde{v}\,(\, \phi(\tilde{u}) + \phi(\tilde{v})\,). \tag{4.1.1f}$$

Before we go through the analysis of the problem $(\widetilde{\mathrm{P}}_M)$, we first demonstrate the point of considering such a problem as an alternative to the model (P). In particular, we clarify the relation between a solution of $(\widetilde{\mathrm{P}}_M)$ and a solution of (P). On noting the system (4.1.1a)-(4.1.1b) and the system (2.2.4a)-(2.2.4b), it can be seen clearly that the problem $(\widetilde{\mathrm{P}}_M)$ is equivalent to (P_M), with $v\,\phi(u)$ replaced by $v\,u$ in (2.2.2), if the number M is chosen large enough such that $\tilde{v} \leq M$. Noting this and the relation between the system (2.2.4a)-(2.2.4b) and the system (1.1.2a)-(1.1.2b), one can deduce the equivalence between the problem $(\widetilde{\mathrm{P}}_M)$ and the problem (P) for M sufficiently large such that $\tilde{u}\,,\, \tilde{v} \leq M$. This equivalence has meaning since the values of u and v, in (P), represent densities of two types of cell populations, which are expected in the biological literature to be bounded (see Painter and Sherratt [55]). We finally mention that our analysis of the problem $(\widetilde{\mathrm{P}}_M)$ will be also restricted to the assumption $D > 0$ as in the analysis of the problem (P_M).

It is convenient to rewrite the system (4.1.1a)-(4.1.1f) in the following multi-component form:

$(\widetilde{\mathbf{P}}_M)$ For fixed $M \geq e$, find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ such that, for $i = 1$ and 2,

$$\frac{\partial u_i}{\partial t} = \nabla \cdot (\, D\, \nabla u_i + \phi(u_i)\, \nabla(u_1 + u_2)) + f_{M,i}(u_1, u_2) \qquad \text{in } \Omega_T, \tag{4.1.2a}$$

$$[\, D\, \nabla u_i + \phi(u_i)\, \nabla(u_1 + u_2)] \cdot \nu = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{4.1.2b}$$

$$u_i(x,0) = u_i^0(x) \qquad \forall\, x \in \Omega, \tag{4.1.2c}$$

where we set $\gamma_1 := 1$ and $\gamma_2 := \gamma > 1$ to define

$$f_{M,i}(u_1, u_2) := \gamma_i\, u_i - u_i\, (\, \phi(u_1) + \phi(u_2)\,). \tag{4.1.2d}$$

The entropy inequality of the problem $(\widetilde{\mathrm{P}}_M)$ can be derived easily by testing the i-th equation in (4.1.2a) with $F'(u_i)$, noting the condition (4.1.2b) and summing over i yielding for $t \in (0,T)$ that

$$\widetilde{E}(t) + \tfrac{D}{M} \int_0^t \sum_{i=1}^{2} \|\nabla u_i\|_0^2\, dt \leq \widetilde{E}(0) + \int_0^t \int_\Omega \sum_{i=1}^{2} f_{M,i}(u_1, u_2)\, F'(u_i)\; \mathrm{d}x\; \mathrm{d}t, \tag{4.1.3}$$

where

$$\widetilde{E}(t) = \int_\Omega \sum_{i=1}^{2} F(u_i)\; \mathrm{d}x\,.$$

As in (2.3.3), the inequality (4.1.3) is only valid for positive functions u_i, $i = 1, 2$. This issue can be efficiently treated by introducing the following regularized problem to $(\widetilde{\mathrm{P}}_M)$:

($\widetilde{\mathbf{P}}_{M,\varepsilon}$) For $M \geq e$ and for $\varepsilon \in (0, e^{-1})$ find $\{u_{\varepsilon,1}(x,t), u_{\varepsilon,2}(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that, for $i = 1$ and 2,

$$\frac{\partial u_{\varepsilon,i}}{\partial t} = \nabla \cdot (\, D\, \nabla u_{\varepsilon,i} + \phi_\varepsilon(u_{\varepsilon,i})\, \nabla(u_{\varepsilon,1} + u_{\varepsilon,2})) + f_{M,i}(u_{\varepsilon,1}, u_{\varepsilon,2}) \quad \text{in } \Omega_T, \tag{4.1.4a}$$

$$[\, D\, \nabla u_{\varepsilon,i} + \phi_\varepsilon(u_{\varepsilon,i})\, \nabla(u_{\varepsilon,1} + u_{\varepsilon,2})] \cdot \nu = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{4.1.4b}$$

$$u_{\varepsilon,i}(x,0) = u_i^0(x) \qquad \forall\, x \in \Omega, \tag{4.1.4c}$$

where

$$f_{M,\varepsilon,i}(u_{\varepsilon,1}, u_{\varepsilon,2}) := \gamma_i\, u_{\varepsilon,i} - u_{\varepsilon,i}\, (\, \phi_\varepsilon(u_{\varepsilon,1}) + \phi_\varepsilon(u_{\varepsilon,2})\,). \tag{4.1.4d}$$

Lemma 4.1.1 Let $u_1^0(x)$ and $u_2^0(x)$ be non-negative bounded functions. There exists a positive $C(u_1^0, u_2^0, M, \gamma)$ independent of ε such that any solution $\{u_{\varepsilon,1}, u_{\varepsilon,2}\}$ of $(\mathrm{P}_{M,\varepsilon})$ satisfies

$$\sup_{0<t<T} \int_\Omega \sum_{i=1}^{2} F_\varepsilon(u_{\varepsilon,i})\ \mathrm{d}x + \tfrac{D}{M} \int_{\Omega_T} \sum_{i=1}^{2} |\nabla u_{\varepsilon,i}|^2\ \mathrm{d}x\, \mathrm{d}t \leq C\,. \tag{4.1.5}$$

Furthermore,

$$\sup_{0<t<T} \int_\Omega \sum_{i=1}^{2} \left(|u_{\varepsilon,i}|^2 + \varepsilon^{-1}\, |[u_{\varepsilon,i}]_-|^2 \right)\ \mathrm{d}x \leq C. \tag{4.1.6}$$

Proof: Testing the i-th equation in (4.1.4a) with $F'_{\varepsilon,i}(u_{\varepsilon,i})$ and summing over i yields on noting (4.1.4b) and (2.3.20) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \sum_{i=1}^{2} F_\varepsilon(u_{\varepsilon,i})\, \mathrm{d}x + D \int_\Omega \sum_{i=1}^{2} \frac{|\nabla u_{\varepsilon,i}|^2}{\phi_\varepsilon(u_{\varepsilon,i})}\ \mathrm{d}x + \int_\Omega |\nabla u_{\varepsilon,1} + \nabla u_{\varepsilon,2}|^2\ \mathrm{d}x \\ = \int_\Omega \sum_{i=1}^{2} f_{M,\varepsilon,i}(u_{\varepsilon,1}, u_{\varepsilon,2})\, F'_\varepsilon(u_{\varepsilon,i})\ \mathrm{d}x\,.\end{aligned} \tag{4.1.7}$$

From (2.3.5), (2.3.8), (2.3.9), (2.1.11), the Young's inequality and (2.3.6) we obtain, for $i = 1$ and 2, that

$$\begin{aligned} f_{M,\varepsilon,i}(u_{\varepsilon,1}, u_{\varepsilon,2})\, F'_\varepsilon(u_{\varepsilon,i}) &\leq \gamma_i\ (2\, F_\varepsilon(u_{\varepsilon,i}) + 1) + (1 - [u_{\varepsilon,i}]_-)\ (\phi_\varepsilon(u_{\varepsilon,1}) + \phi_\varepsilon(u_{\varepsilon,2})) \\ &\leq 2\, \gamma_i\, F_\varepsilon(u_{\varepsilon,i}) + \tfrac{1}{2\,\varepsilon}\, [u_{\varepsilon,i}]_-^2 + \tfrac{\varepsilon}{2}\ (\phi_\varepsilon(u_{\varepsilon,1}) + \phi_\varepsilon(u_{\varepsilon,2}))^2 + C(M, \gamma_i) \\ &\leq (2\, \gamma_i + 1)\, F_\varepsilon(u_{\varepsilon,i}) + C(M, \gamma_i)\,. \end{aligned} \tag{4.1.8}$$

Dropping the third term in (4.1.7) and noting (4.1.8) and (2.3.5) gives that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \sum_{i=1}^{2} F_\varepsilon(u_{\varepsilon,i})\,\mathrm{d}x + \tfrac{D}{M}\int_\Omega \sum_{i=1}^{2} |\nabla u_{\varepsilon,i}|^2\,\mathrm{d}x \le C(M,\gamma_i) + \int_\Omega \sum_{i=1}^{2} (2\,\gamma_i + 1)\,F_\varepsilon(u_{\varepsilon,i})\,\mathrm{d}x\,. \tag{4.1.9}$$

Hence, the desired result (4.1.5) follows immediately from (4.1.9) after application of the Grönwall lemma and recalling the initial condition (4.1.4c). The result (4.1.6) follows from (4.1.9), (2.3.6) and (2.3.7). □

4.2 A fully discrete finite element approximation

Let the assumptions (A) hold. For any $\varepsilon \in (0, e^{-1})$, we consider the following fully discrete finite element approximation of $(\widetilde{\mathrm{P}}_{M,\varepsilon})$:

$(\widetilde{\mathbf{P}}^{h,\Delta t}_{M,\varepsilon})$ For $n \ge 1$ find $\{U^n_{\varepsilon,1}, U^n_{\varepsilon,2}\} \in S^h \times S^h$ such that for $i = 1$ and 2, and for all $\chi \in S^h$

$$\left(\tfrac{U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}}{\Delta t_n}, \chi\right)^h + \left(D\,\nabla U^n_{\varepsilon,i} + \Lambda_\varepsilon(U^n_{\varepsilon,i})\,\nabla\left(U^n_{\varepsilon,1} + U^n_{\varepsilon,2}\right), \nabla\chi\right) = \left(\gamma_i\, U^n_{\varepsilon,i} - U^n_{\varepsilon,i}\,\left[\phi_\varepsilon(U^{n-1}_{\varepsilon,1}) + \phi_\varepsilon(U^{n-1}_{\varepsilon,2})\right], \chi\right)^h, \tag{4.2.1}$$

where $U^0_{\varepsilon,i} \in S^h$, for $i = 1$ and 2, is an approximation of u^0_i.

Theorem 4.2.1 Let the assumptions (A) hold, $D > 0$, $\gamma_1 = 1$ and $\gamma_2 > 1$. Let $\{U^{n-1}_{\varepsilon,1}, U^{n-1}_{\varepsilon,2}\} \in S^h \times S^h$ be a given solution to the $(n-1)$-th step of $(\widetilde{\mathrm{P}}^{h,\Delta t}_{M,\varepsilon})$ for some $n = 1, \cdots, N$. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ and for all $\Delta t_n > 0$ such that $\Delta t_n \le \frac{1}{2\,\gamma+1}$, there exists a solution $\{U^n_{\varepsilon,1}, U^n_{\varepsilon,2}\} \in S^h \times S^h$ to the n-th step of $(\widetilde{\mathrm{P}}^{h,\Delta t}_{M,\varepsilon})$ satisfying

$$(1 - (2\,\gamma + 1)\,\Delta t_n)\sum_{i=1}^{2} (F_\varepsilon(U^n_{\varepsilon,i}), 1)^h + \tfrac{D}{M}\sum_{i=1}^{2} \Delta t_n \left|U^n_{\varepsilon,i}\right|_1^2 \le \sum_{i=1}^{2} (F_\varepsilon(U^{n-1}_{\varepsilon,i}), 1)^h + C\,\Delta t_n\,. \tag{4.2.2}$$

Proof: The existence proof is a simple modification of the proof of Theorem 2.4.7. We now sketch the proof of (4.2.2). Choosing $\chi \equiv \Delta t_n\, \pi^h[F'_\varepsilon(U^n_{\varepsilon,i})]$ as a test function in the i-th equation in (4.2.1) and noting (2.4.22b) and (2.4.3) yields, for $i = 1$ and 2, that

$$\left(U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}, F'_\varepsilon(U^n_{\varepsilon,i})\right)^h + \Delta t_n \left(D\, \left[\Lambda_\varepsilon(U^n_{\varepsilon,i})\right]^{-1} \nabla U^n_{\varepsilon,i} + \nabla(U^n_{\varepsilon,1} + U^n_{\varepsilon,2}), \nabla U^n_{\varepsilon,i}\right)$$
$$= \Delta t_n \left(\gamma_i\, U^n_{\varepsilon,i} - U^n_{\varepsilon,i}\, \left[\phi_\varepsilon(U^{n-1}_{\varepsilon,1}) + \phi_\varepsilon(U^{n-1}_{\varepsilon,2})\right], F'_\varepsilon(U^n_{\varepsilon,i})\right)^h, \qquad (4.2.3)$$

which is a discrete analogue of (4.1.7).

It follows from (2.3.5), (2.3.8), (2.3.9), (2.1.11), the Young's inequality and (2.3.6), for $i = 1$ and 2, that

$$\begin{aligned}\Delta t_n &\left(\gamma_i\, U^n_{\varepsilon,i} - U^n_{\varepsilon,i}\, \left[\phi_\varepsilon(U^{n-1}_{\varepsilon,1}) + \phi_\varepsilon(U^{n-1}_{\varepsilon,2})\right], F'_\varepsilon(U^n_{\varepsilon,i})\right)^h \\
&\leq \gamma_i\, \Delta t_n\, (2\, F_\varepsilon(U^n_{\varepsilon,i}) + 1, 1)^h + \Delta t_n \left(\phi_\varepsilon(U^{n-1}_{\varepsilon,1}) + \phi_\varepsilon(U^{n-1}_{\varepsilon,2}), 1\right)^h \\
&\qquad - \Delta t_n \left(\phi_\varepsilon(U^{n-1}_{\varepsilon,1}) + \phi_\varepsilon(U^{n-1}_{\varepsilon,2}), [U^n_{\varepsilon,i}]_-\right)^h \\
&\leq 2\, \gamma_i \Delta t_n\, (F_\varepsilon(U^n_{\varepsilon,i}), 1)^h + \tfrac{\Delta t_n}{2\,\varepsilon}\, \left|[U^n_{\varepsilon,i}]_-\right|^2_h + C(M,\, \gamma_i\,,\, |\Omega|)\, \Delta t_n \\
&\leq (2\, \gamma_i + 1)\, \Delta t_n\, (F_\varepsilon(U^n_{\varepsilon,i}), 1)^h + C(M,\, \gamma_i\,,\, |\Omega|)\, \Delta t_n\,. \end{aligned} \qquad (4.2.4)$$

Combining (4.2.3), (4.2.4) and (2.4.58a) yields, for $i = 1$ and 2, that

$$(1 - (2\, \gamma_i\, + 1)\, \Delta t_n)\, (F_\varepsilon(U^n_{\varepsilon,i}), 1)^h + \Delta t_n \left(D\, \left[\Lambda_\varepsilon(U^n_{\varepsilon,i})\right]^{-1} \nabla U^n_{\varepsilon,i} + \nabla(U^n_{\varepsilon,1} + U^n_{\varepsilon,2}), \nabla U^n_{\varepsilon,i}\right)$$
$$\leq (F_\varepsilon(U^{n-1}_{\varepsilon,i}), 1)^h + C\, \Delta t_n\,. \quad (4.2.5)$$

Thus, the result (4.2.2) follows by summing (4.2.5) over i and noting (2.4.32), $F_\varepsilon(s) \geq 0$ and that $\gamma > 1$. $\square$

In the following theorem we derive discrete analogues of the estimates obtained in Lemma 4.1.1.

Theorem 4.2.2 Let the assumptions of Theorem 4.2.1 hold and let $u^0_i \in L^\infty(\Omega)$ with $u^0_i(x) \geq 0$ for a.e. $x \in \Omega$, $i = 1\,, 2$. Let either $U^0_{\varepsilon,i} \equiv P^h u^0_i$; or $U^0_{\varepsilon,i} \equiv \pi^h u^0_i$ if $u^0_i \in C(\overline{\Omega})$. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ and for all $\Delta t > 0$ such that $\Delta t \leq \frac{1-\delta}{2\,\gamma+1}$, for some $\delta \in (0,1)$, the problem $(\widetilde{\mathrm{P}}^{h,\Delta t}_{M,\varepsilon})$ possesses a solution $\{U^n_{\varepsilon,1}, U^n_{\varepsilon,2}\}_{n=1}^N$ such that

$$\max_{n=1,\cdots,N} \sum_{i=1}^{2} \left[(F_\varepsilon(U^n_{\varepsilon,i}),1)^h + \varepsilon^{-1}\|\pi^h[U^n_{\varepsilon,i}]_-\|_0^2 + \|U^n_{\varepsilon,i}\|_0^2\right] + \sum_{n=1}^{N} \Delta t_n \sum_{i=1}^{2} \|U^n_{\varepsilon,i}\|_1^2 \leq C\,. \tag{4.2.6}$$

Furthermore,

$$\sum_{n=1}^{N} \Delta t_n \sum_{i=1}^{2} \left[\|U^n_{\varepsilon,i}\|_{0,\alpha}^{\alpha} + \|\tfrac{U^n_{\varepsilon,i}-U^{n-1}_{\varepsilon,i}}{\Delta t_n}\|^2_{(H^1(\Omega))'} + \|\mathcal{G}[\tfrac{U^n_{\varepsilon,i}-U^{n-1}_{\varepsilon,i}}{\Delta t_n}]\|_1^2\right] \leq C\,, \tag{4.2.7}$$

where $\alpha = \frac{2\,(d+2)}{d}$.

Proof: It follows from (4.2.2) and the assumption on Δt, for $n = 1, \cdots, N$, that

$$\begin{aligned}\sum_{i=1}^{2}(F_\varepsilon(U^n_{\varepsilon,i}),1)^h &\leq \left(1 + \tfrac{(2\,\gamma+1)\,\Delta t_n}{\delta}\right)\sum_{i=1}^{2}(F_\varepsilon(U^{n-1}_{\varepsilon,i}),1)^h + \tfrac{C}{\delta}\,\Delta t_n \\ &\leq e^{\frac{(2\,\gamma+1)\,\Delta t_n}{\delta}} \sum_{i=1}^{2}(F_\varepsilon(U^{n-1}_{\varepsilon,i}),1)^h + \tfrac{C}{\delta}\,\Delta t_n\,.\end{aligned} \tag{4.2.8}$$

The first bound in (4.2.6) follows from (4.2.8) and the assumptions on the initial data. The second and the third bounds in (4.2.6) follow directly from the first bound in (4.2.6) on recalling (2.4.2), (2.3.6) and (2.3.7). The last bound in (4.2.6) follows by summing (4.2.2) over n and noting the first and the third bounds in (4.2.6).

Similarly to (3.2.15), it follows from (2.1.1) and the third bound in (4.2.6) for $i = 1\,,2$, for $n = 1, \cdots, N$ and for the stated choice of α that

$$\|U^n_{\varepsilon,i}\|_{0,\alpha}^{\alpha} \leq C\,\|U^n_{\varepsilon,i}\|_0^{\alpha-2}\|U^n_{\varepsilon,i}\|_1^2 \leq C\,\|U^n_{\varepsilon,i}\|_1^2\,. \tag{4.2.9}$$

We obtain from (3.1.8), (3.1.1), (4.2.1), (2.4.2), (2.3.5), (2.4.24), (2.4.25), (3.1.3) and (4.2.6) for any $\eta \in H^1(\Omega)$, for $i = 1\,,2$ and for $n = 1, \cdots, N$ that

$$\begin{aligned}\langle \tfrac{U^n_{\varepsilon,i}-U^{n-1}_{\varepsilon,i}}{\Delta t_n}, \eta\rangle &= \left(\tfrac{U^n_{\varepsilon,i}-U^{n-1}_{\varepsilon,i}}{\Delta t_n}, \eta\right) = \left(\tfrac{U^n_{\varepsilon,i}-U^{n-1}_{\varepsilon,i}}{\Delta t_n}, P^h\eta\right)^h \\ &= (\gamma_i\,U^n_{\varepsilon,i} - U^n_{\varepsilon,i}\,\left[\phi_\varepsilon(U^{n-1}_{\varepsilon,1}) + \phi_\varepsilon(U^{n-1}_{\varepsilon,2})\right], P^h\eta)^h \\ &\qquad - \left(D\,\nabla U^n_{\varepsilon,i} + \Lambda_\varepsilon(U^n_{\varepsilon,i})\,\nabla(U^n_{\varepsilon,1}+U^n_{\varepsilon,2}), \nabla P^h\eta\right) \\ &\leq C\,\|U^n_{\varepsilon,i}\|_0\,\|P^h\eta\|_0 + C\,\left(|U^n_{\varepsilon,1}|_1 + |U^n_{\varepsilon,2}|_1\right)\,|P^h\eta|_1 \\ &\leq C\,\|\eta\|_1 \sum_{i=1}^{2}\|U^n_{\varepsilon,i}\|_1\,,\end{aligned}$$

which implies

$$\|\tfrac{U^n_{\varepsilon,i}-U^{n-1}_{\varepsilon,i}}{\Delta t_n}\|^2_{(H^1(\Omega))'} \leq C \sum_{i=1}^{2} \|U^n_{\varepsilon,i}\|^2_1 \,. \tag{4.2.10}$$

Combining (4.2.9), (4.2.10), (3.1.9) and the last bound in (4.2.6) provides the desired result (4.2.7). $\square$

Remark 4.2.1 We note that the proof of Theorem 4.2.2 does not require non-increasing time-step discretizations, while such a condition is essential in the proof of Theorem 3.2.3. This is due to the nature of the discretization of the reaction term in the approximation problem (4.2.1).

4.3 Convergence and existence results

To prove existence of a global weak solution of the system (4.1.2a)-(4.1.2d), we pass to the limit $\varepsilon\,,\,h\,,\,\Delta t\ \to 0$ of the approximation system (4.2.1). For that purpose, we first need to adapt the notation (3.3.1a)-(3.3.3a) to $U_{\varepsilon,i}$, $i = 1$ and 2 , and restate the problem $(\widetilde{\mathrm{P}}^{h,\Delta t}_{M,\varepsilon})$ as follows:

Find $\{U_{\varepsilon,1},\, U_{\varepsilon,2}\} \in C([0,T];S^h) \times C([0,T];S^h)$ such that for $i = 1$ and 2 , and for all $\chi \in L^2(0,T;S^h)$

$$\int_0^T \left[\left(\tfrac{\partial U_{\varepsilon,i}}{\partial t}, \chi\right)^h + D\,(\,\nabla U^+_{\varepsilon,i}, \nabla\chi) + (\,\Lambda_\varepsilon(U^+_{\varepsilon,i})\,\nabla(U^+_{\varepsilon,1}+U^+_{\varepsilon,2}), \nabla\chi) \right]\,\mathrm{d}t$$
$$= \int_0^T \left(\gamma_i\, U^+_{\varepsilon,i} - U^+_{\varepsilon,i}\,\left[\phi_\varepsilon(U^-_{\varepsilon,1}) + \phi_\varepsilon(U^-_{\varepsilon,2})\right], \chi\right)^h \mathrm{d}t\,. \tag{4.3.1}$$

Theorem 4.3.1 Let the assumptions (A) hold, $D > 0$, $\gamma_1 = 1$, $\gamma_2 > 1$ and $u^0_i \in H^1_{\geq 0}(\Omega) \cap L^\infty(\Omega)$, $i = 1$ and 2 . In addition, let $\left\{\, \varepsilon, h, \{\Delta t_n\}_{n=1}^N, U^0_{\varepsilon,1}, U^0_{\varepsilon,2} \,\right\}$ be such that

(i) either $U^0_{\varepsilon,i} \equiv P^h u^0_i$; or $U^0_{\varepsilon,i} \equiv \pi^h u^0_i$ if either $d = 1$ or $u^0_i \in W^{1,r}(\Omega)$ with $r > d$, $i = 1, 2$.

(ii) $\Delta t \leq \frac{1-\delta}{2\,\gamma+1}$, for some $\delta \in (0,1)$.

(iii) Δt , $\varepsilon \ \to 0$ as $h \ \to 0$.

Then there exists a subsequence of $\{U_{\varepsilon,1}, U_{\varepsilon,2}\}_{h>0}$, where $\{U_{\varepsilon,1}, U_{\varepsilon,2}\}$ solves (4.3.1), and functions

$$u_i \in L^2(0,T;H^1(\Omega)) \cap L^\alpha(\Omega_T) \cap L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;(H^1(\Omega))')\,, \quad (4.3.2a)$$

where $\alpha = \frac{2\,(d+2)}{d}$, with

$$u_i(x,t) \geq 0 \;\; a.e. \text{ in } \Omega_T \quad \text{and} \quad u_i(\cdot,0) = u_i^0(\cdot) \;\text{ in } L^2(\Omega) \qquad i = 1\,,\; 2\,. \quad (4.3.2b)$$

Moreover, it holds as $h \to 0$ that

$$
\begin{aligned}
U_{\varepsilon,i},\; U_{\varepsilon,i}^{\pm} &\rightharpoonup u_i && \text{in } \; L^2(0,T;H^1(\Omega)) \cap L^\alpha(\Omega_T)\,, && (4.3.3a)\\
U_{\varepsilon,i},\; U_{\varepsilon,i}^{\pm} &\rightharpoonup^* u_i && \text{in } \; L^\infty(0,T;L^2(\Omega))\,, && (4.3.3b)\\
\frac{\partial U_{\varepsilon,i}}{\partial t} &\rightharpoonup \frac{\partial u_i}{\partial t} && \text{in } \; L^2(0,T;(H^1(\Omega))')\,, && (4.3.3c)\\
U_{\varepsilon,i},\; U_{\varepsilon,i}^{\pm} &\to u_i && \text{in } \; L^2(0,T;L^s(\Omega))\,, && (4.3.3d)\\
\phi_\varepsilon(U_{\varepsilon,i}^{\pm}) &\to \phi(u_i) && \text{in } \; L^2(0,T;L^s(\Omega))\,, && (4.3.3e)\\
\pi^h\phi_\varepsilon(U_{\varepsilon,i}^{\pm}) &\to \phi(u_i) && \text{in } \; L^2(0,T;L^s(\Omega))\,, && (4.3.3f)\\
\Lambda_\varepsilon(U_{\varepsilon,i}^{\pm}) &\to \phi(u_i)\,\mathcal{I} && \text{in } \; L^2(0,T;L^s(\Omega))\,, && (4.3.3g)
\end{aligned}
$$

for any

$$s \in \begin{cases} [2,\infty] & \text{if } d = 1\,,\\ [2,\infty) & \text{if } d = 2\,,\\ [2,6) & \text{if } d = 3\,. \end{cases}$$

Proof: Similarly to (3.3.8a), it follows from (4.2.6), (4.2.7), (2.3.5), (2.4.24), (2.4.25) and our assumptions on the initial data for $i = 1$ and 2 that

$$
\begin{aligned}
&\|U_{\varepsilon,i}^{(\pm)}\|_{L^2(0,T;H^1(\Omega))} + \|U_{\varepsilon,i}^{(\pm)}\|_{L^\alpha(\Omega_T)} + \|U_{\varepsilon,i}^{(\pm)}\|_{L^\infty(0,T;L^2(\Omega))}\\
&\quad + \varepsilon^{-\frac{1}{2}}\|\pi^h[U_{\varepsilon,i}^{(\pm)}]_-\|_{L^\infty(0,T;L^2(\Omega))} + \|\tfrac{\partial U_{\varepsilon,i}}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')}\\
&\qquad + \|\mathcal{G}\tfrac{\partial U_{\varepsilon,i}}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\phi_\varepsilon(U_{\varepsilon,i}^{(\pm)})\|_{L^\infty(\Omega_T)}\\
&\qquad\quad + \|\pi^h\phi_\varepsilon(U_{\varepsilon,i}^{(\pm)})\|_{L^\infty(\Omega_T)} + \|\Lambda_\varepsilon(U_{\varepsilon,i}^{(\pm)})\|_{L^\infty(\Omega_T)} \leq C\,. \qquad (4.3.4)
\end{aligned}
$$

On noting the uniform bounds in (4.3.4), the proof of the theorem can be easily established by following exactly the same arguments used to show the results concerning $U_\varepsilon^{(\pm)}$ and u in Theorem 3.3.1. □

Theorem 4.3.2 Let the assumptions of Theorem 4.3.1 hold. Then there exists a subsequence of $\{U_{\varepsilon,1},\, U_{\varepsilon,2}\}_{h>0}$, where $\{U_{\varepsilon,1},\, U_{\varepsilon,2}\}$ solves $(\widetilde{\mathrm{P}}^{h,\Delta t}_{M,\varepsilon})$, and non-negative functions $\{u_1, u_2\}$ satisfying (4.3.2a)-(4.3.2b). In addition, as $h \to 0$ the convergence results (4.3.3a)-(4.3.3g) hold. Furthermore, the functions $\{u_1, u_2\}$ represent a global weak solution of the problem $(\widetilde{\mathrm{P}}_M)$ in sense that for $i = 1\,,\, 2$

$$\int_0^T \big[\big\langle \tfrac{\partial u_i}{\partial t}, \eta \big\rangle + D\,(\,\nabla u_i, \nabla \eta) + (\,\phi(u_i)\,\nabla(u_1+u_2), \nabla \eta)\,\big]\,\mathrm{d}t$$
$$= \int_0^T (\gamma_i\, u_i - u_i\,(\,\phi(u_1)+\phi(u_2)\,), \eta)\;\mathrm{d}t \qquad \forall \eta \in L^2(0,T;H^1(\Omega))\,. \quad (4.3.5)$$

Proof: The first and the second parts of the theorem follow from Theorem 4.3.1. To show that the functions $\{u_1, u_2\}$ satisfy (4.3.5), we set $\chi \equiv \pi^h \eta$ as a test function in (4.3.1) and then we pass to the limit $\varepsilon\,,\, h\,,\, \Delta t \;\to 0\,.$ The procedure is similar to the proof of (3.3.28a) in Theorem 3.3.3. □

Remark 4.3.1 On recalling Remark 3.3.3, one can consider the problem $(\widetilde{\mathrm{P}}_M)$ for $M \in [1, e)$ and use (3.3.71), instead of (2.3.8), to obtain the same results achieved for $M \geq e\,.$ In this case, the restrictions on the time-discretization parameter in Theorem 4.2.1 and Theorem 4.2.2 are replaced by the conditions

$$\Delta t_n \leq \tfrac{2}{5\gamma+2} \qquad \text{and} \qquad \Delta t \leq \tfrac{2\,(1-\delta)}{5\,\gamma+2}\,, \quad \text{for some } \delta \in (0,1)\,,$$

respectively. Clearly, for $\gamma < 2$, these restrictions are weaker than the corresponding restrictions mentioned in Remark 3.3.3 for the problem (P_M).

The following observation is related to the reaction terms in the problem (P):

Remark 4.3.2 It is worth mentioning that one can consider more general reaction terms in the problem (P) similar to those considered in the model studied in [21] and [9]. Namely, we can easily adapt the analysis presented in our previous work for studying the problem (GP) where (GP) is the same as (P) but with $f(u,v)$ in (1.1.2a) replaced by $u\,(\,\gamma_1 - \mu_{1,1}\,u - \mu_{1,2}\,v\,)$ and $g(u,v)$ in (1.1.2b) replaced by $v\,(\,\gamma_2 - \mu_{2,1}\,u - \mu_{2,2}\,v\,)\,,\; \mu_{i,j}\,,\, \gamma_i \,\geq\, 0$ for $i, j \,=\, 1$ and $2\,.$ The following section is devoted to the discussion of some additional properties of the solutions of the model

(P) in the absence of competition between the two types of cell population. This is when $\mu_{i,j}$ and γ_i , $i, j = 1$ and 2, in the problem (GP) are all equal to zero; see the model (1.1.1a)-(1.1.1b).

4.4 The population model with no reaction terms

In this section we consider the following cross diffusion model representing the dynamics of two cell populations:

(P$_0$) Find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ such that, for $i = 1$ and 2,

$$\frac{\partial u_i}{\partial t} = \nabla \cdot (\, D\,\nabla u_i + u_i\,\nabla(u_1+u_2)) \qquad \text{in } \Omega_T, \tag{4.4.1a}$$

$$[\, D\,\nabla u_i + u_i\,\nabla(u_1+u_2)] \cdot \nu = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{4.4.1b}$$

$$u_i(x,0) = u_i^0(x) \qquad \forall\, x \in \Omega. \tag{4.4.1c}$$

Clearly, in one dimension space the above model represents the model (1.1.1a)-(1.1.1b). Here, the constant D is also assumed to be strictly positive. As discussed in Section 4.1, the key of the analysis of the system (4.4.1a)-(4.4.1c) is to consider an alternative system:

($\widetilde{\mathbf{P}}_{0,M}$) For fixed $M \geq 1$, find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ such that, for $i = 1$ and 2,

$$\frac{\partial u_i}{\partial t} = \nabla \cdot (\, D\,\nabla u_i + \phi(u_i)\,\nabla(u_1+u_2)) \qquad \text{in } \Omega_T, \tag{4.4.2a}$$

$$[\, D\,\nabla u_i + \phi(u_i)\,\nabla(u_1+u_2)] \cdot \nu = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{4.4.2b}$$

$$u_i(x,0) = u_i^0(x) \qquad \forall\, x \in \Omega, \tag{4.4.2c}$$

where $\phi(u_i) := \phi_M(u_i)$ is defined by (2.2.1).

Theorem 4.4.1 Let the assumptions (A) hold, $D > 0$ and $u_i^0 \in H^1_{\geq 0}(\Omega) \cap L^\infty(\Omega)$, $i = 1$ and 2. Then for any $T > 0$ there exists a global in-time weak solution $\{u_1, u_2\}$ of the system (4.4.2a)-(4.4.2c) satisfying (4.3.2a)-(4.3.2b) and $⨍\, u_i = ⨍\, u_i^0$ for *a.e.* $t \in (0,T)$, $i = 1$ and 2, such that for all $\eta \in L^2(0,T; H^1(\Omega))$

$$\int_0^T \left[\left\langle \tfrac{\partial u_i}{\partial t}, \eta \right\rangle + D\,(\,\nabla u_i, \nabla\eta) + (\,\phi(u_i)\,\nabla(u_1+u_2), \nabla\eta)\,\right]\,\mathrm{d}t = 0\,, \tag{4.4.3}$$

where the mean integral used above is defined for any integrable function ω by

$$\fint \omega := \frac{1}{|\Omega|}\,(\omega, 1)\,.$$

Furthermore, if

$$0 < l \leq u_i^0(x) \leq M \quad \text{in } \ \Omega \qquad i = 1 \text{ and } 2\,, \tag{4.4.4}$$

then there exist constants $C_0(M\,,\, u_1^0\,,\, u_2^0) \geq 0$ and $C_*(D\,,\, C_p) > 0$ such that

$$\int_0^T \sum_{i=1}^{2} \|u_i - \fint u_i^0\|_0^2 \,\mathrm{d}t \ \leq \tfrac{C_0}{C_*}\left(1 - e^{-C_* T}\right). \tag{4.4.5}$$

Proof: The existence proof can be easily established similarly to Theorem 4.3.2 where we consider, under the assumptions (A), for any $\varepsilon \in (0, e^{-1})$ the following fully discrete finite element approximation of $(\widetilde{\mathrm{P}}_{0,M})$:

$(\widetilde{\mathbf{P}}^{h,\Delta t}_{0,M,\varepsilon})$ For $n \geq 1$ find $\{U^n_{\varepsilon,1}, U^n_{\varepsilon,2}\} \in S^h \times S^h$ such that for $i = 1$ and 2, and for all $\chi \in S^h$

$$\left(\tfrac{U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}}{\Delta t_n}, \chi\right)^h + \left(D\,\nabla U^n_{\varepsilon,i} + \Lambda_\varepsilon(U^n_{\varepsilon,i})\,\nabla\left(U^n_{\varepsilon,1} + U^n_{\varepsilon,2}\right), \nabla\chi\right) = 0\,, \tag{4.4.6}$$

where $U^0_{\varepsilon,i} \in S^h$ is an appropriate approximation of u_i^0.

Obviously, choosing $\eta \equiv 1$ in (4.4.3) and noting (4.4.2c) yields for *a.e.* $t \in (0, T)$ that

$$\fint u_i(t) = \fint u_i(0) = \fint u_i^0 \qquad i = 1 \text{ and } 2\,. \tag{4.4.7}$$

Now, we consider the result (4.4.5) which explains, in some sense, the long time behaviour of the derived solution of $(\widetilde{\mathrm{P}}_{0,M})$. Assume that (4.4.4) is satisfied. It is convenient for our purpose to define, for $i = 1$ and 2 and for any $\varepsilon \in (0, l)$, the function $\widetilde{F}_{\varepsilon,i} : \mathbb{R}^{\geq 0} \to \mathbb{R}$ such that

$$\widetilde{F}_{\varepsilon,i}(s) := \begin{cases} \frac{s^2 - \varepsilon^2}{2\,\varepsilon} + \left(\ln\varepsilon - \ln\left(\fint u_i^0\right) - 1\right)\, s + \fint u_i^0 & \text{if } s \leq \varepsilon\,, \\[2ex] \left(\ln s - \ln\left(\fint u_i^0\right) - 1\right)\, s + \fint u_i^0 & \text{if } \varepsilon \leq s \leq M\,, \\[2ex] \frac{s^2 - M^2}{2\,M} + \left(\ln M - \ln\left(\fint u_i^0\right) - 1\right)\, s + \fint u_i^0 & \text{if } s \geq M\,. \end{cases} \tag{4.4.8a}$$

Hence $\widetilde{F}_{\varepsilon,i}(s) \in C^2(\mathbb{R}^{\geq 0})$, $i = 1$ and 2 , with the first two derivatives of $\widetilde{F}_{\varepsilon,i}(s)$ given, respectively, by

$$\widetilde{F}'_{\varepsilon,i}(s) = F'_\varepsilon(s) - \ln\left(⨍ \, u_i^0\right) \tag{4.4.8b}$$

$$\text{and} \qquad \widetilde{F}''_{\varepsilon,i}(s) = F''_\varepsilon(s)\,. \tag{4.4.8c}$$

We obtain from a Taylor expansion around $⨍ \, u_i^0$, on noting (4.4.8a)-(4.4.8c) and (4.4.4) that for $i = 1$ and 2

$$\begin{aligned} \widetilde{F}_{\varepsilon,i}(s) &= \widetilde{F}_{\varepsilon,i}(⨍ \, u_i^0) + \left(s - ⨍ \, u_i^0\right) \, \widetilde{F}'_{\varepsilon,i}(⨍ \, u_i^0) + \tfrac{1}{2} \, \left(s - ⨍ \, u_i^0\right)^2 \, F''_\varepsilon(\xi_i) \\ &\geq \tfrac{1}{2\,M} \, \left(s - ⨍ \, u_i^0\right)^2 . \end{aligned} \tag{4.4.9}$$

Testing the i-th equation in (4.4.3) with $\eta \equiv \widetilde{F}'_{\varepsilon,i}(u_i) \in L^2(0,T;H^1(\Omega))$, as $u_i \in L^2(0,T;H^1(\Omega))$, and summing the resulting equations yields on noting (2.3.5) that

$$\begin{aligned} \sum_{i=1}^{2} \left(\widetilde{F}_{\varepsilon,i}(u_i(T)), 1\right) + \int_0^T \sum_{i=1}^{2} \left[D \, (F''_\varepsilon(u_i)\, \nabla u_i, \nabla u_i) + \left(\frac{\phi(u_i)}{\phi_\varepsilon(u_i)} \, \nabla(u_1 + u_2), \nabla u_i\right)\right] \, \mathrm{d}t \\ = \sum_{i=1}^{2} \left(\widetilde{F}_{\varepsilon,i}(u_i^0), 1\right) \, . \end{aligned} \tag{4.4.10}$$

It follows from (4.4.10), (4.4.9) and (2.3.4c) that

$$\begin{aligned} \sum_{i=1}^{2} \tfrac{1}{2\,M} \, \|u_i(T) - ⨍ \, u_i^0\|_0^2 + \tfrac{D}{M} \int_0^T \sum_{i=1}^{2} |u_i|_1^2 \, \mathrm{d}t + \int_{\Omega_T} \left(\sum_{i=1}^{2} \left(\frac{\phi(u_i)}{\phi_\varepsilon(u_i)}\right)^{\frac{1}{2}} \nabla u_i\right)^2 \mathrm{d}x \, \mathrm{d}t \\ \leq \sum_{i=1}^{2} \left(\widetilde{F}_{\varepsilon,i}(u_i^0), 1\right) + \int_{\Omega_T} \left| \left(\frac{\phi(u_1)}{\phi_\varepsilon(u_1)}\right)^{\frac{1}{2}} - \left(\frac{\phi(u_2)}{\phi_\varepsilon(u_2)}\right)^{\frac{1}{2}} \right|^2 |\nabla u_1| \, |\nabla u_2| \, \mathrm{d}x \, \mathrm{d}t \, . \end{aligned} \tag{4.4.11}$$

The assumption (4.4.4) allows us to set

$$C_0 := 2\,M \sum_{i=1}^{2} \left(\widetilde{F}_{\varepsilon,i}(u_i^0), 1\right) = 2\,M \sum_{i=1}^{2} \left(\ln u_i^0 - \ln\left(⨍ \, u_i^0\right) \, , \, u_i^0\right) \, . \tag{4.4.12}$$

Neglecting the third term in the left hand side of (4.4.11) and letting $\varepsilon \to 0$ yields, on noting (4.4.12), that

$$\sum_{i=1}^{2} \|u_i(T) - ⨍ \, u_i^0\|_0^2 + 2\,D \int_0^T \sum_{i=1}^{2} |u_i|_1^2 \, \mathrm{d}t \, \leq C_0 \, . \tag{4.4.13}$$

As $u_i \in H^1(\Omega)$, $i = 1$ and 2, we obtain from the Poincaré inequality (2.1.8) and (4.4.7) that

$$\|u_i - \fint u_i^0\|_0^2 \leq C_p\,|u_i|_1^2 \qquad i = 1 \text{ and } 2\,. \tag{4.4.14}$$

Substituting (4.4.14) into (4.4.13) leads to

$$\sum_{i=1}^{2} \|u_i(T) - \fint u_i^0\|_0^2 + 2\,D\,C_p^{-1} \int_0^T \sum_{i=1}^{2} \|u_i - \fint u_i^0\|_0^2 \;\mathrm{d}t \;\leq C_0\,. \tag{4.4.15}$$

On setting $C_* := 2\,D\,C_p^{-1}$, the desired result (4.4.5) follows from (4.4.15) after simple calculations. $\square$

Chapter 5

The population model: Numerical experiments

This chapter is devoted to the discussion of some numerical solutions for the model (P) in one space dimension. We introduce an iterative approach to solve our fully discrete finite element approximation to problem (P_M). We then establish and discuss some numerical solutions for different choices of the parameters γ, D and M. We also introduce a modified iterative scheme to obtain the numerical solutions of problem $(\widetilde{\mathrm{P}}_M)$. Hence, we make an experimental comparison between the solutions of (P_M) and $(\widetilde{\mathrm{P}}_M)$. In addition, we obtain and discuss some other numerical results. We use programs written in Fortran, see Appendix B.1, to generate the numerical results and Matlab to plot the graphs.

5.1 Numerical results

We first introduce the following practical algorithm to solve the nonlinear algebraic system arising from the approximate problem $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ at each time level:
Given $\{U_\varepsilon^{n,0},\ V_\varepsilon^{n,0}\} \in S^h \times S^h$, for $k \geq 1$ find $\{U_\varepsilon^{n,k}, V_\varepsilon^{n,k}\} \in S^h \times S^h$ such that for

all $\chi \in S^h$

$$\left(\tfrac{U_\varepsilon^{n,k}-U_\varepsilon^{n-1}}{\Delta t_n},\chi\right)^h + (D\,\nabla U_\varepsilon^{n,k} + \Lambda_\varepsilon(U_\varepsilon^{n,k-1})\,\nabla(U_\varepsilon^{n,k}+V_\varepsilon^{n,k}),\nabla\chi)$$
$$= (U_\varepsilon^{n,k} - U_\varepsilon^{n,k}\,\phi_\varepsilon(U_\varepsilon^{n-1}) - \phi_\varepsilon(U_\varepsilon^{n,k-1})\,\psi_\varepsilon(V_\varepsilon^{n-1}),\chi)^h\,, \tag{5.1.1a}$$
$$\left(\tfrac{V_\varepsilon^{n,k}-V_\varepsilon^{n-1}}{\Delta t_n},\chi\right)^h + (D\,\nabla V_\varepsilon^{n,k} + \Xi_\varepsilon(V_\varepsilon^{n,k-1})\,\nabla(V_\varepsilon^{n,k}+U_\varepsilon^{n,k}),\nabla\chi)$$
$$= (\gamma\,V_\varepsilon^{n,k} - \psi_\varepsilon(V_\varepsilon^{n,k-1})\,[\phi_\varepsilon(U_\varepsilon^{n-1}) + \psi_\varepsilon(V_\varepsilon^{n-1})]\,,\chi)^h\,, \tag{5.1.1b}$$

where we start with $U_\varepsilon^0 \equiv \pi^h u^0$ and $V_\varepsilon^0 \equiv \pi^h v^0$, and we set, for $n \geq 1$, $U_\varepsilon^{n,0} \equiv U_\varepsilon^{n-1}$ and $V_\varepsilon^{n,0} \equiv V_\varepsilon^{n-1}$. As the system (5.1.1a)-(5.1.1b) is linear, existence of $\{U^{n,k}, V^{n,k}\}$ follows from uniqueness. The latter can be easily investigated on noting (2.4.32) and (2.4.33). The standard method to solve the system (5.1.1a)-(5.1.1b) at each iteration is by testing the equations (5.1.1a) and (5.1.1b) with $\varphi_j,\ j = 0,\cdots,J$, to obtain a $(2\,J+2)\times(2\,J+2)$ linear system, in terms of the nodal values of $U^{n,k}$ and $V^{n,k}$, which can be solved using linear programming. For our numerical results, we set $TOL = 1\times 10^{-7}$ and adopt the stopping criteria

$$\left|U_\varepsilon^{n,k} - U_\varepsilon^{n,k-1}\right|_{0,\infty} < TOL \quad \text{and} \quad \left|V_\varepsilon^{n,k} - V_\varepsilon^{n,k-1}\right|_{0,\infty} < TOL, \tag{5.1.2}$$

i.e. for k satisfying (5.1.2) we set $U_\varepsilon^n \equiv U_\varepsilon^{n,k}$ and $V_\varepsilon^n \equiv V_\varepsilon^{n,k}$. We have been unable to prove convergence of $\{U^{n,k}, V^{n,k}\}_{k=1}^\infty$ to $\{U^n, V^n\}$ for n fixed. However, in practice we found that the iterative method always converged well (only a few steps were required to fulfill the stopping criteria at each time level).

We now present some numerical results in one space dimension. Unless otherwise specified, in all experiments we consider a uniform partitioning of $\Omega = (0,5)$ into 256 subintervals, (i.e. $J = 256$ and $h = \frac{5}{256}$), and choose $\Delta t_n = \Delta t = 10^{-3}$, $n \geq 1$, and $\varepsilon = 10^{-9}$.

In the first part of our experiments, we considered the dynamics of two interacting cell populations which were initially distributed symmetrically in the domain via the initial conditions $u^0(x) = \frac{1}{2} + \frac{1}{2}\cos(\frac{4\pi}{5}\,x)$ and $v^0(x) = \frac{1}{2} - \frac{1}{2}\cos(\frac{4\pi}{5}\,x)$. We took the parameters $D = 1$ and $M = 1$. To illustrate how the parameter γ could reflect a competitive advantage of the v cells over the u cells, we performed the experiment firstly for $\gamma = 1$ and secondly for $\gamma = 2$. The numerical solutions of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ are plotted in Figure 5.1(a)-(b) at several times. These times are chosen carefully to demonstrate the evolution of the interacting cells as t increases. We observed that for sufficiently large time the solution reaches a steady state. In the case $\gamma = 1$, the cells evolve to form a homogeneous distribution; see Figure 5.1(a). The same behaviour is observed when $\gamma = 2$, but with a distinguished advantage of the v cells; see Figure 5.1(b).

We repeated the above experiment for $D = 100$. The general behaviour was the same, but the stationary solutions were achieved earlier for $\gamma = 1$ and later for $\gamma = 2$; see Figure 5.2(a)-(b).

In the previous experiments the total cell density was initially constant, namely $u^0(x) + v^0(x) = 1$, and hence each population moves down its own gradient as claimed in [55]. Furthermore, we note that due to the large diffusivity in the case $D = 100$, the movement to the direction of lower concentrations is faster than the case when $D = 1$.

To show the effect of the terms $\nabla \cdot [u\nabla(u + v)]$ and $\nabla \cdot [v\nabla(u + v)]$ which are imposed in (P) to ensure that cells move down gradients in the total density, we chose $u^0(x) = -0.2\,x + 1$ and $v^0(x) = 1$. Here the u cells are initially seeded with a gradient in the cell density while the v cells are seeded at a uniform density. We took $D = 1$ and $M = 1$. The numerical solutions are plotted in Figure 5.3(a)-(b) for $\gamma = 1$ and $\gamma = 1.5$, respectively. The cells move to the direction of lower total density, before both cell types become homogeneously distributed. This agrees with the biological point of view explained in the introduction of problem (P).

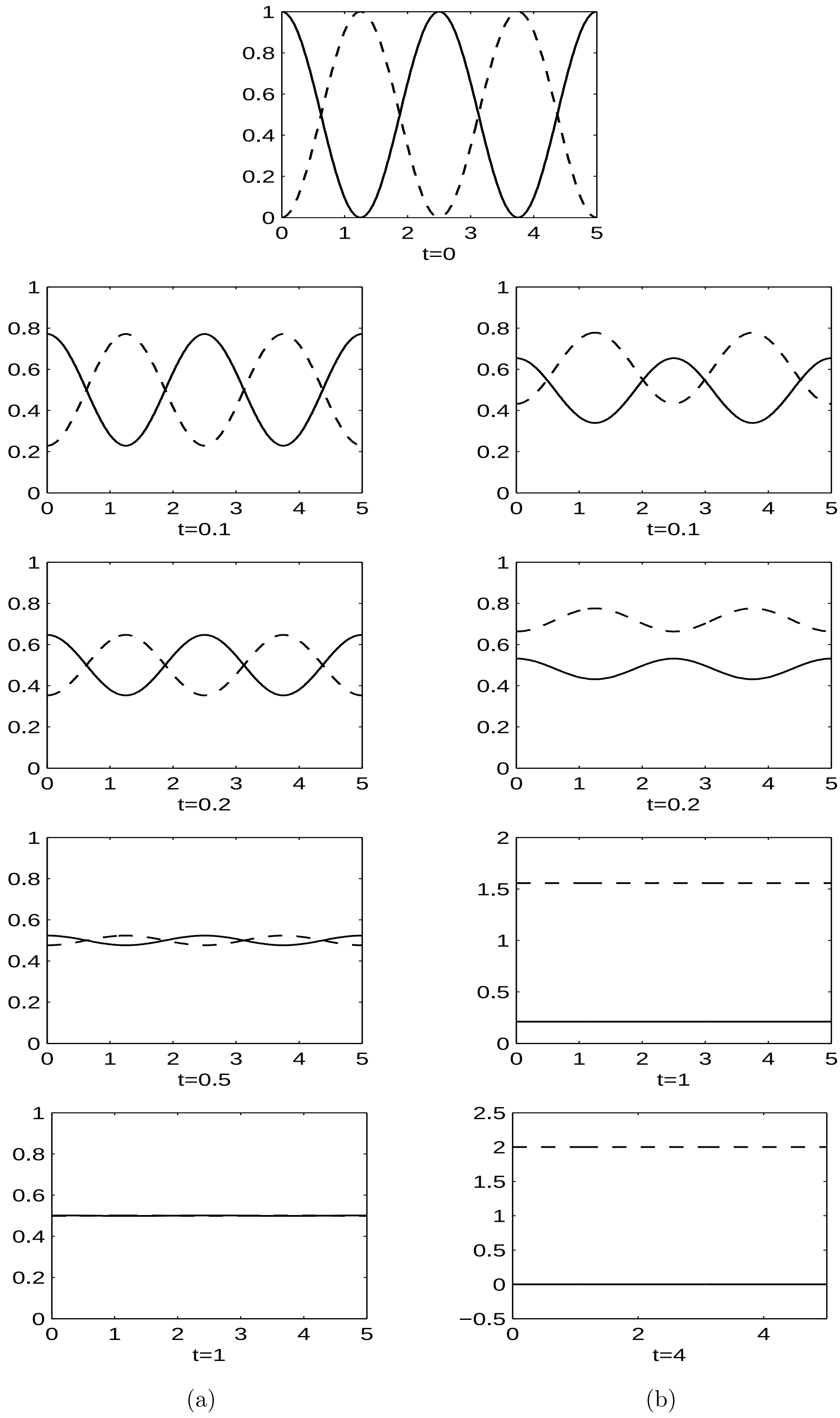
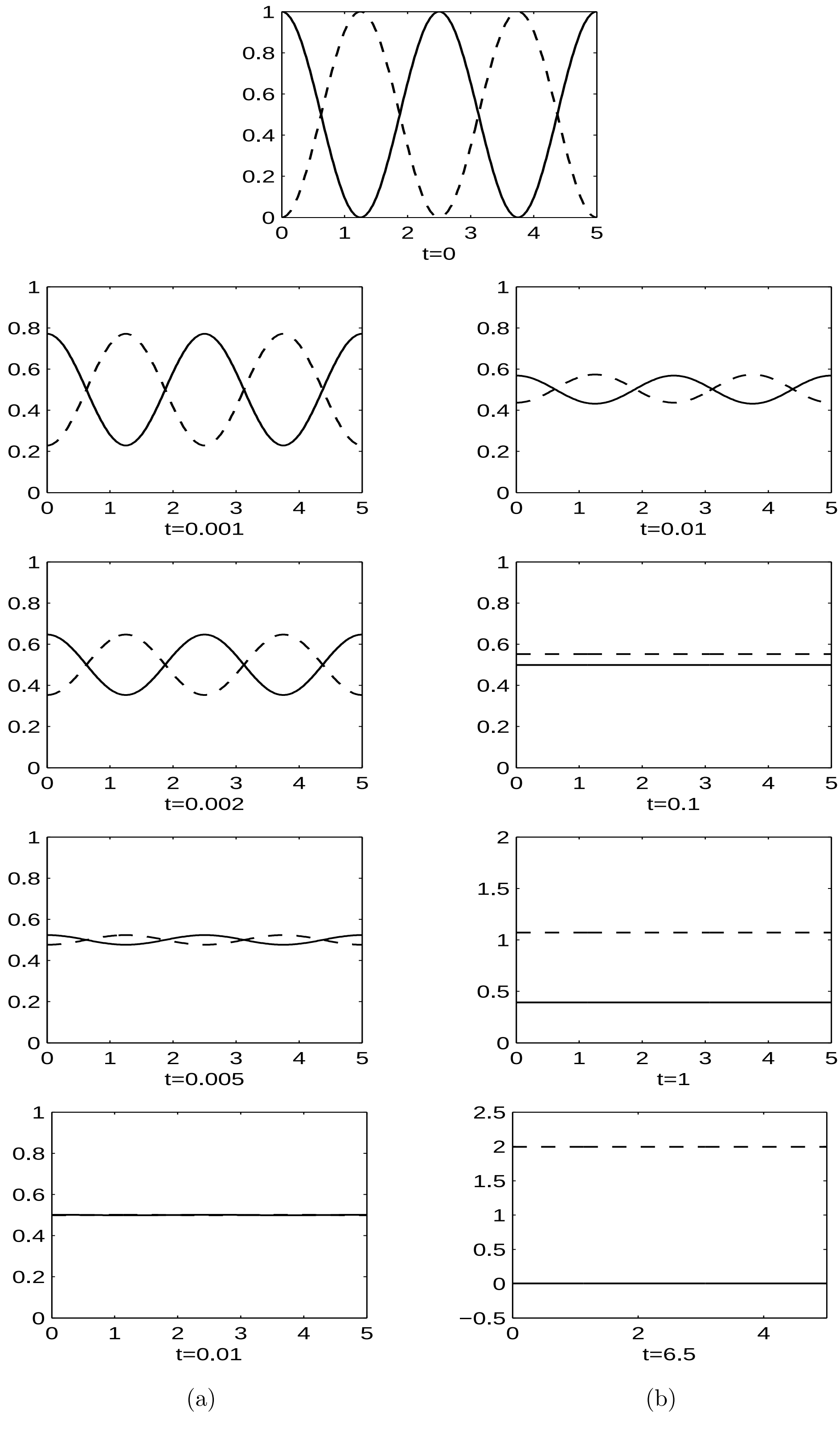


Figure 5.1: Numerical solutions of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ plotted at several times. The initial data are $u^0(x) = \frac{1}{2} + \frac{1}{2}\cos(\frac{4\pi}{5}x)$ and $v^0(x) = \frac{1}{2} - \frac{1}{2}\cos(\frac{4\pi}{5}x)$. The parameter values are: $D = 1$, $M = 1$, with $\gamma = 1$ in (a) and $\gamma = 2$ in (b). The solid and dashed lines represent u and v, respectively.



(a) (b)

Figure 5.2: Numerical solutions of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ plotted at several times. The initial data are $u^0(x) = \frac{1}{2} + \frac{1}{2}\cos(\frac{4\pi}{5}\,x)$ and $v^0(x) = \frac{1}{2} - \frac{1}{2}\cos(\frac{4\pi}{5}\,x)$. The parameter values are: $D = 100$, $M = 1$, with $\gamma = 1$ in (a) and $\gamma = 2$ in (b). The solid and dashed lines represent u and v, respectively.

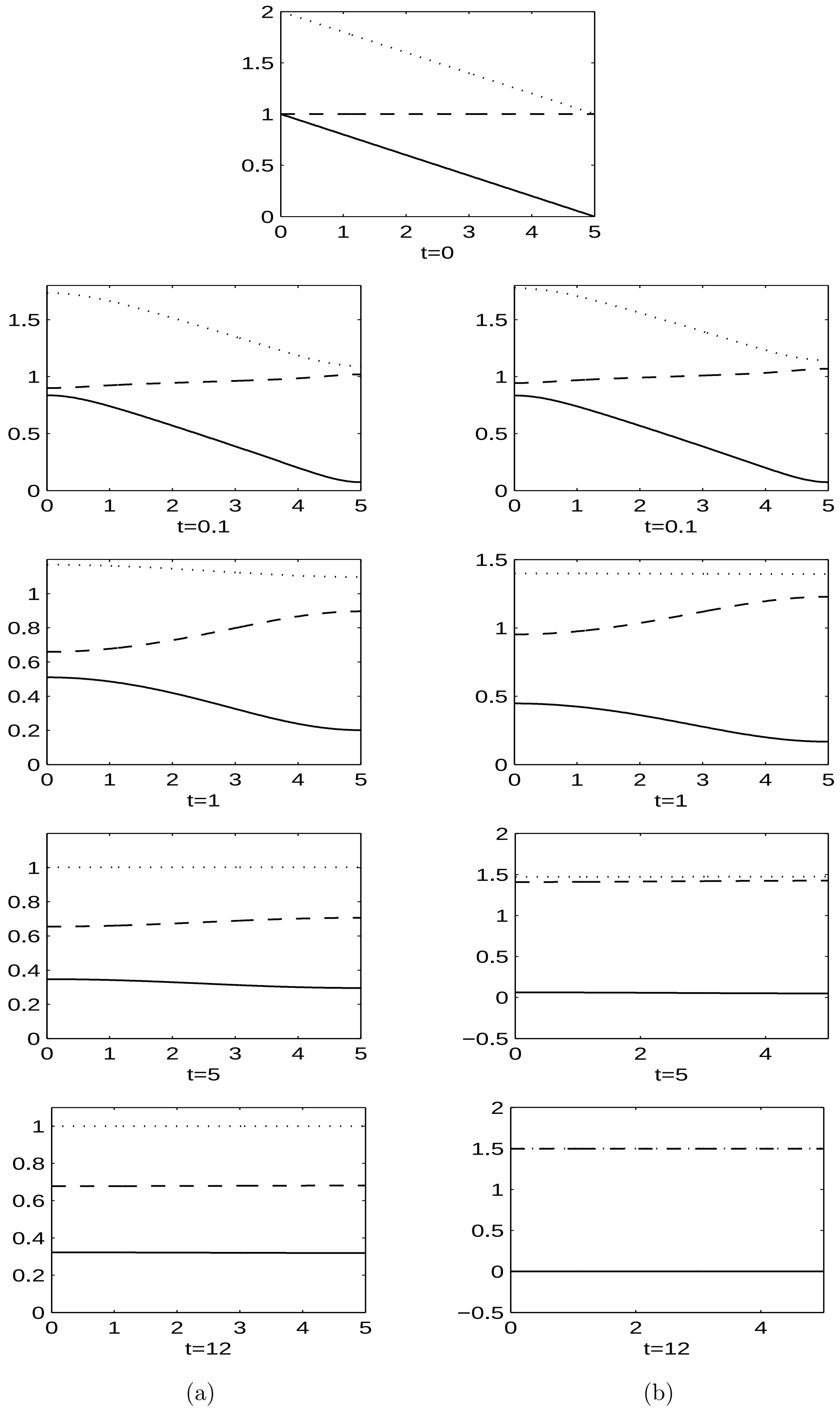


Figure 5.3: Numerical solutions of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ plotted at several times. The initial data are $u^0(x) = -0.2\,x + 1$ and $v^0(x) = 1$. The parameter values are: $D = 1$, $M = 1$, with $\gamma = 1$ in (a) and $\gamma = 1.5$ in (b). The solid, dashed and dotted lines represent u, v and $u+v$, respectively.

In all our previous experiments, the computed solution for u did not exceed the value M. Therefore, based on the discussion in Section 2.2, the established solutions of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ can be considered as numerical approximations of problem (P). We also report that repeating these experiments for different values of $M > 1$, leads us to obtain the same results.

We note that the steady-state solution of (P) in space and time, denoted by $\{u_c, v_c\}$, is determined by the following equations

$$u_c\,(1 - u_c - v_c) = 0,$$
$$v_c\,(\gamma - u_c - v_c) = 0.$$

For $\gamma > 1$, the u cells will vanish in (P) due to the advantage of the v cells; and hence we should expect to have $u_c = 0$ and $v_c = \gamma$. For $\gamma = 1$, we clearly have either $u_c = v_c = 0$ or $u_c + v_c = 1$. This is satisfied by all numerical steady-state solutions in our experiments.

In the following experiments, we see how different choices of the parameter M might lead us to obtain different solutions. For this purpose, we considered a "non-realistic" situation where we choose the initial data $u^0(x) = x$ and $v^0(x) = 1$, with the parameters $D = 1$ and $\gamma = 4$. The solutions corresponding to the values $M = 1$, 2 and 10 are plotted in Figure 5.4 at several times labeled with M values. Since $\gamma = 4 > 1$, the same steady states are approached for the values $M = 1$, 2 and 10. However, this is not the case in the absence of the competitive advantage, i.e. when $\gamma = 1$; see Figure 5.5. If we are seeking a solution to problem (P), we should consider the one obtained using the parameter value $M = 10$ as in this case the model (P_M) is equivalent to (P). Numerically, repeating the experiment with any choice $M \geq 5$ would give the same results for $M = 10$. Again, this is because the computed solution for u does not exceed the value 5 at any stage of evolution.

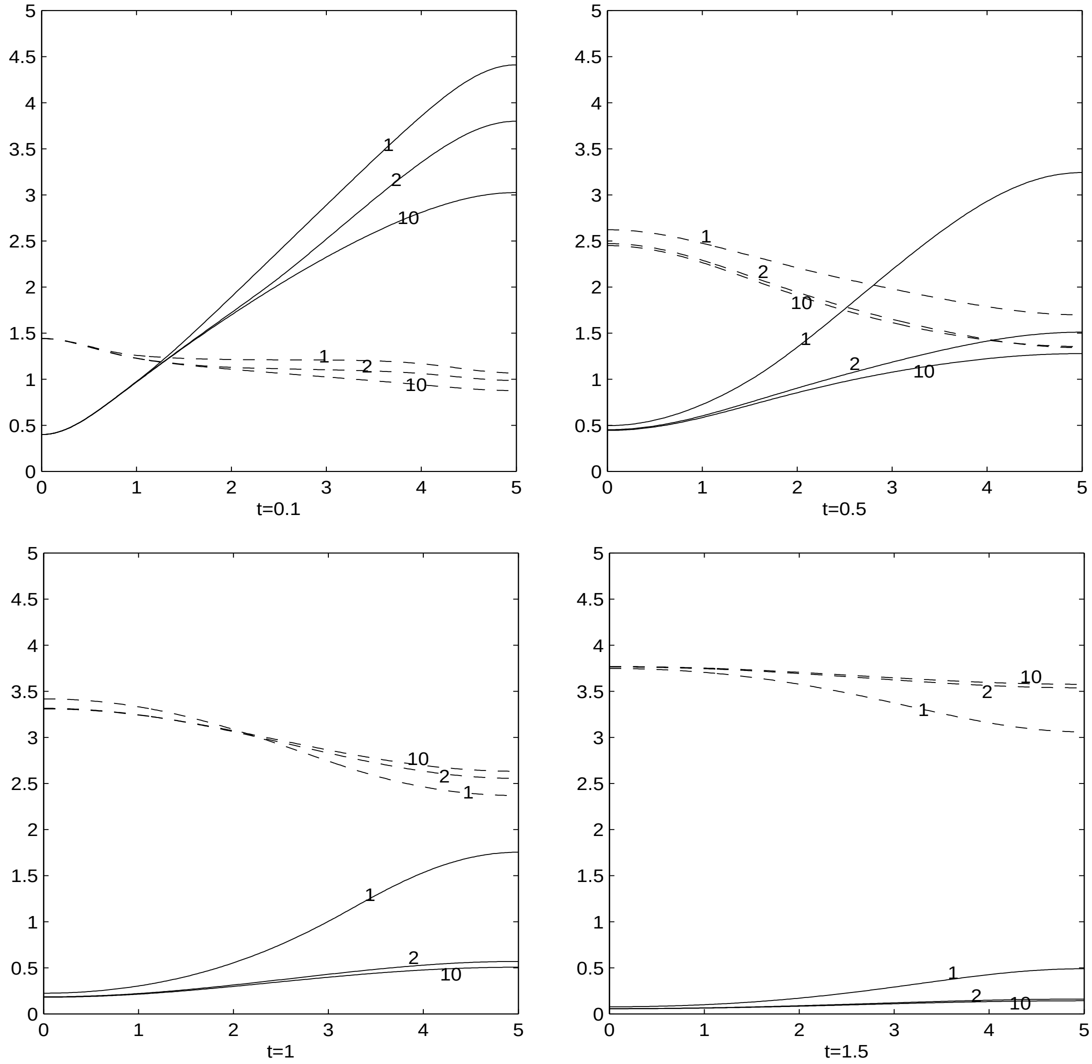
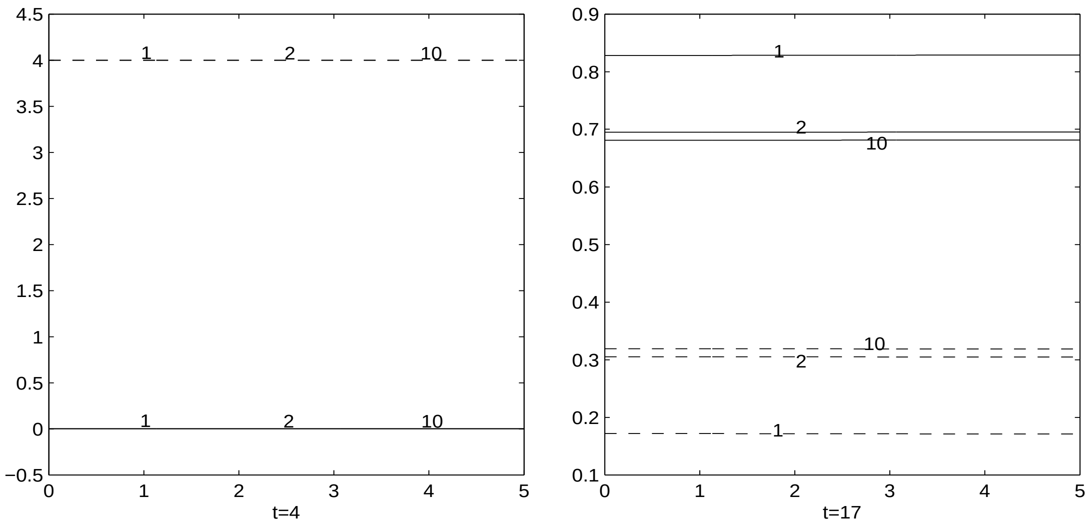


Figure 5.4: Numerical solutions of $(\mathrm{P}^{h,\Delta t}_{M,\varepsilon})$, for different values of M, plotted at several times. The initial data are $u^0(x) = x$ and $v^0(x) = 1$. The parameters are: $D = 1$, $\gamma = 4$ and $M = 1$, 2 and 10. The lines are labelled with M values where the solid lines represent u and the dashed lines represent v.



(a) (b)

Figure 5.5: Numerical steady-state solutions of $(\mathrm{P}^{h,\Delta t}_{M,\varepsilon})$ plotted for $M = 1$, 2 and 10. The initial data are $u^0(x) = x$ and $v^0(x) = 1$. The parameters are: $D = 1$ with (a) $\gamma = 4$, (b) $\gamma = 1$. The lines are labelled with M values where the solid lines represent u and the dashed lines represent v.

A natural question is: How to choose an appropriate value of M that leads to a numerical solution to (P)? The practical answer is a simple matter as one can initially start with a value M which satisfies $||U^0_\varepsilon||_{0,\infty} \leq M$ and adopt the following criterion in the solver: For fixed n and k, if $||U^{n,k}_\varepsilon||_{0,\infty} > M$ then set $M = ||U^{n,k}_\varepsilon||_{0,\infty}$ and recompute $\{U^{n,k}_\varepsilon, V^{n,k}_\varepsilon\}$.

It has been observed through the theoretical analysis of problem (P) that the assumption $D > 0$ is essential to obtain the stability bounds which are required to conclude the convergence results. However, in the next experiment, we have repeated the experiments shown in Figure 5.1 and Figure 5.3 for $D = 0$. This is the case when cells respond only to the total density gradient. The solutions are presented in Figure 5.6 and Figure 5.7 respectively. Since the total density in Figure 5.6 is constant, no movement occurs in the case $\gamma = 1$ and the cells remain in the initial state; see Figure 5.6(a). In fact, in the absence of the diffusion and cross diffusion terms, one easily can show that the system: Find $\{u(x,t), v(x,t)\}$ such that

$$\begin{aligned}
\frac{\partial u}{\partial t} &= u\,(1-u-v)\,,\\
\frac{\partial v}{\partial t} &= v\,(1-u-v)\,,\\
u^0(x) &= \tfrac{1}{2}+\tfrac{1}{2}\,\cos(\tfrac{4\pi}{5}\,x),\\
v^0(x) &= \tfrac{1}{2}-\tfrac{1}{2}\,\cos(\tfrac{4\pi}{5}\,x),
\end{aligned}$$

has the solution $\{u^0(x), v^0(x)\}$ which is independent of t. Expectedly, the presence of the competitive advantage in Figure 5.6(b) caused movement since the initial distribution of u develops into a sharp cell aggregation before it eventually vanishes due to the domination of the v cells. In Figure 5.7, mixing occurs until the total cell density becomes homogeneous. However, the individual densities may remain inhomogeneously mixed; cf. Figure 5.7(a). This is agreed with the observations in [55].

The rapid changes of the solutions in Figure 5.6(b) is a point of interest. As an attempt to investigate whether such behaviour is due to the existence of a singularity when $D = 0$, we have repeated the experiment in Figure 5.6(b) for $D = 0.1,\ 0.01,\ 0.001,\ 0.0001$ and 0 with a finer mesh (we took $h = \frac{5}{1024}$). The solutions $U_\varepsilon(\cdot, 2)$ and $V_\varepsilon(\cdot, 2)$ are plotted in Figure 5.8(a)-(b) respectively. As D decreases to zero, the solutions change rapidly at $x = 0,\ 2.5$ and 5. The solutions appear to be continuous but we expect there will be limited regularity when $D = 0$, i.e. $u, v \notin C^{0,1}$. We also note that the solutions behave smoothly outside the small neighborhoods of $x = 0, 2.5$ and 5. It may be possible in future work to investigate the behaviour of the solution around points of rapid change by performing small-parameter expansions (see the techniques used in [16]).

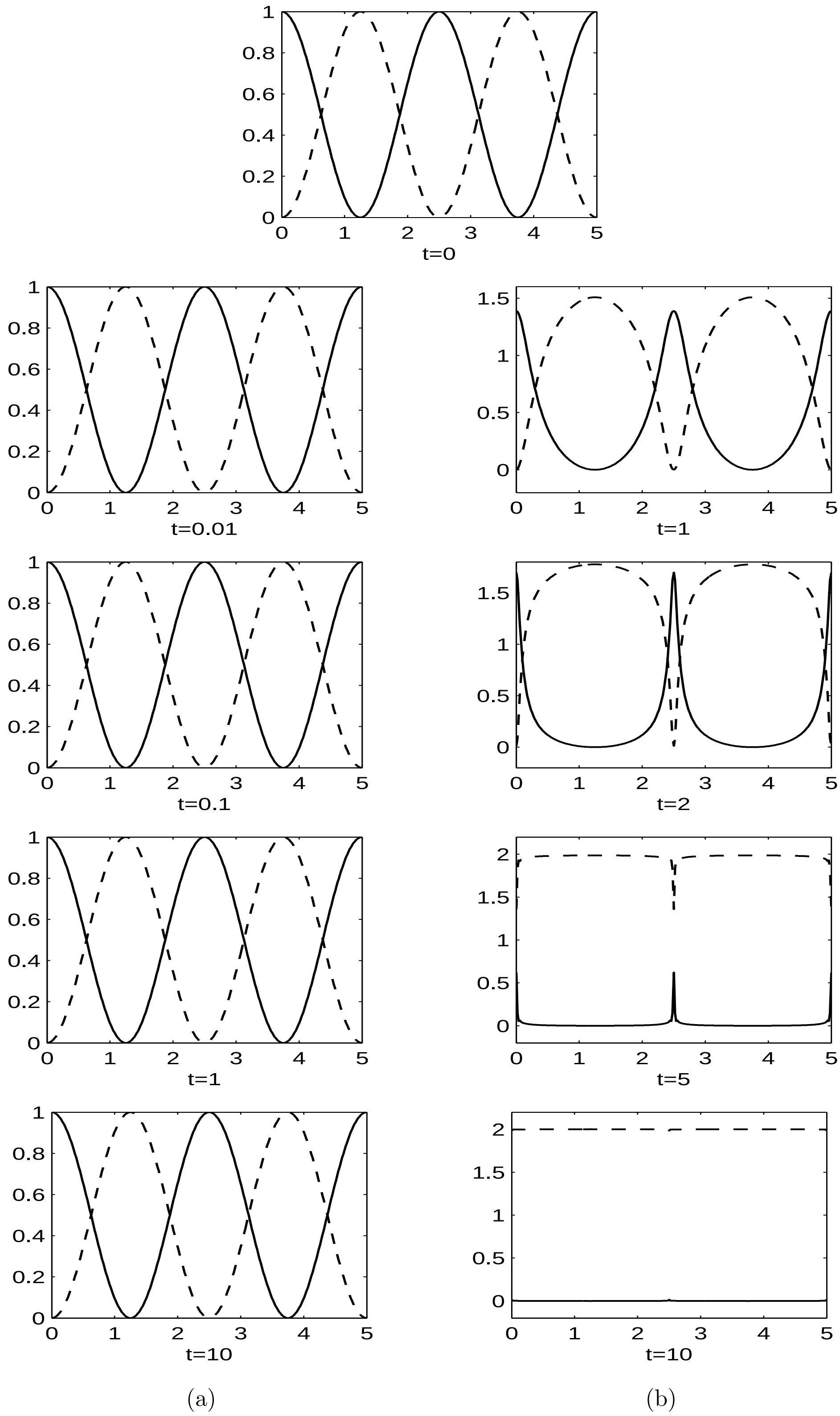


Figure 5.6: Numerical solutions of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ plotted at several times. The initial data are $u^0(x) = \frac{1}{2} + \frac{1}{2}\cos(\frac{4\pi}{5}\,x)$ and $v^0(x) = \frac{1}{2} - \frac{1}{2}\cos(\frac{4\pi}{5}\,x)$. The parameter values are: $D = 0$, $M = 10$, with $\gamma = 1$ in (a) and $\gamma = 2$ in (b). The solid and dashed lines represent u and v, respectively.

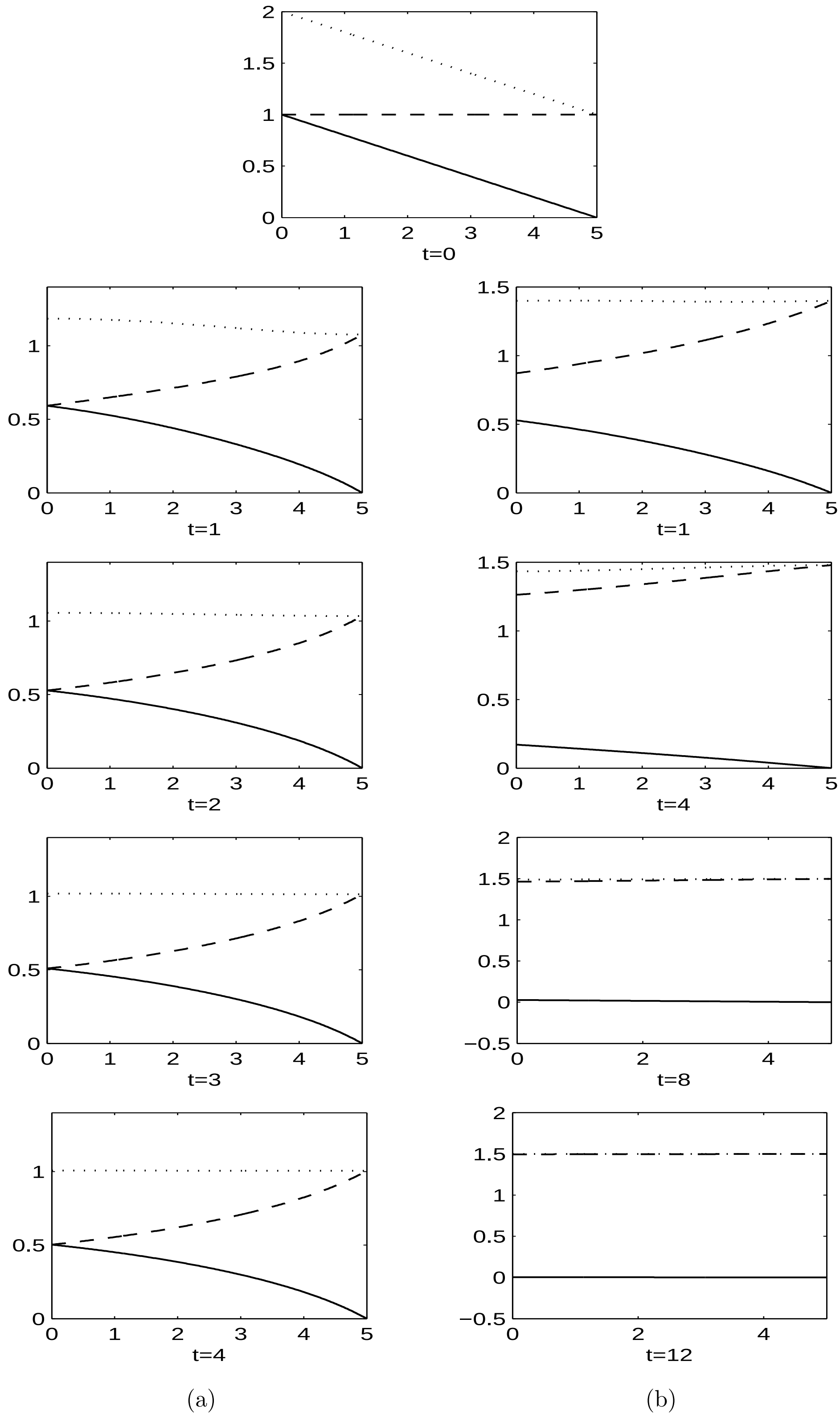
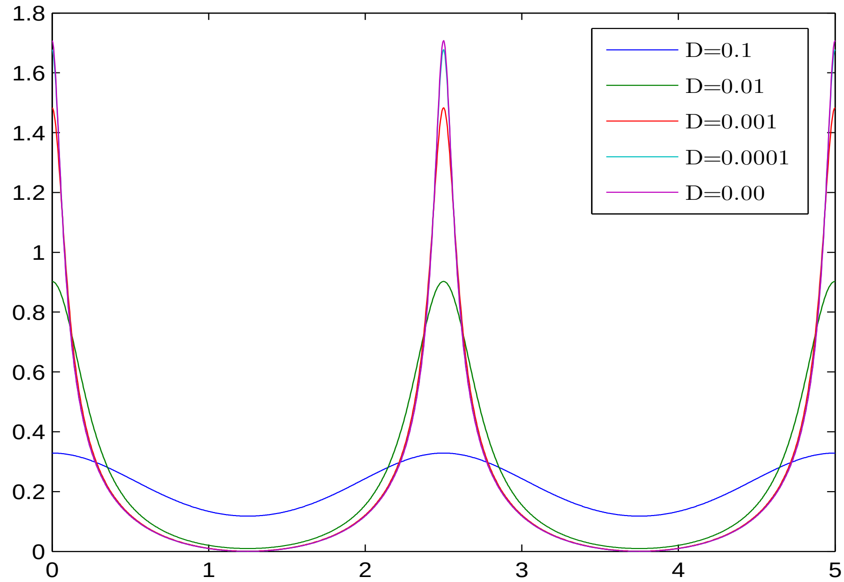
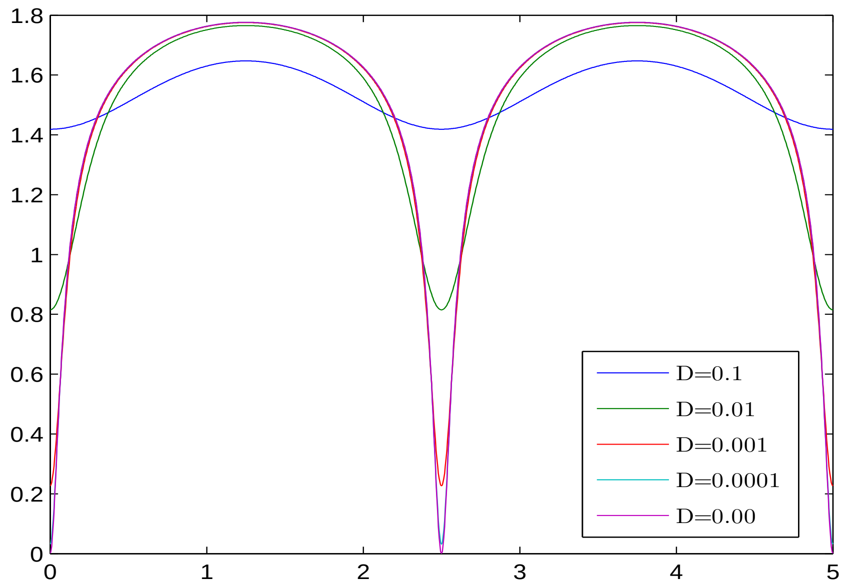


Figure 5.7: Numerical solutions of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ plotted at several times. The initial data are $u^0(x) = -0.2\,x + 1$ and $v^0(x) = 1$. The parameter values are: $D = 0$, $M = 1$, with $\gamma = 1$ in (a) and $\gamma = 1.5$ in (b). The solid, dashed and dotted lines represent u, v and $u + v$, respectively.



(a) $U_\varepsilon(\cdot, 2)$



(b) $V_\varepsilon(\cdot, 2)$

Figure 5.8: Numerical solutions of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ plotted at time $t = 2$. The initial data are $u^0(x) = \frac{1}{2} + \frac{1}{2}\cos(\frac{4\pi}{5}x)$ and $v^0(x) = \frac{1}{2} - \frac{1}{2}\cos(\frac{4\pi}{5}x)$. The solutions are plotted for different parameter values of D, with $M = 10$ and $\gamma = 2$.

Problem $(\widetilde{\mathrm{P}}_M)$:

Naturally, the iterative algorithm (5.1.1a)-(5.1.1b) can be modified to obtain numerical solutions of the finite element approximation $(\widetilde{\mathrm{P}}^{h,\Delta t}_{M,\varepsilon})$. Namely, we propose the following iterative scheme to solve the system (4.2.1) at each time level:

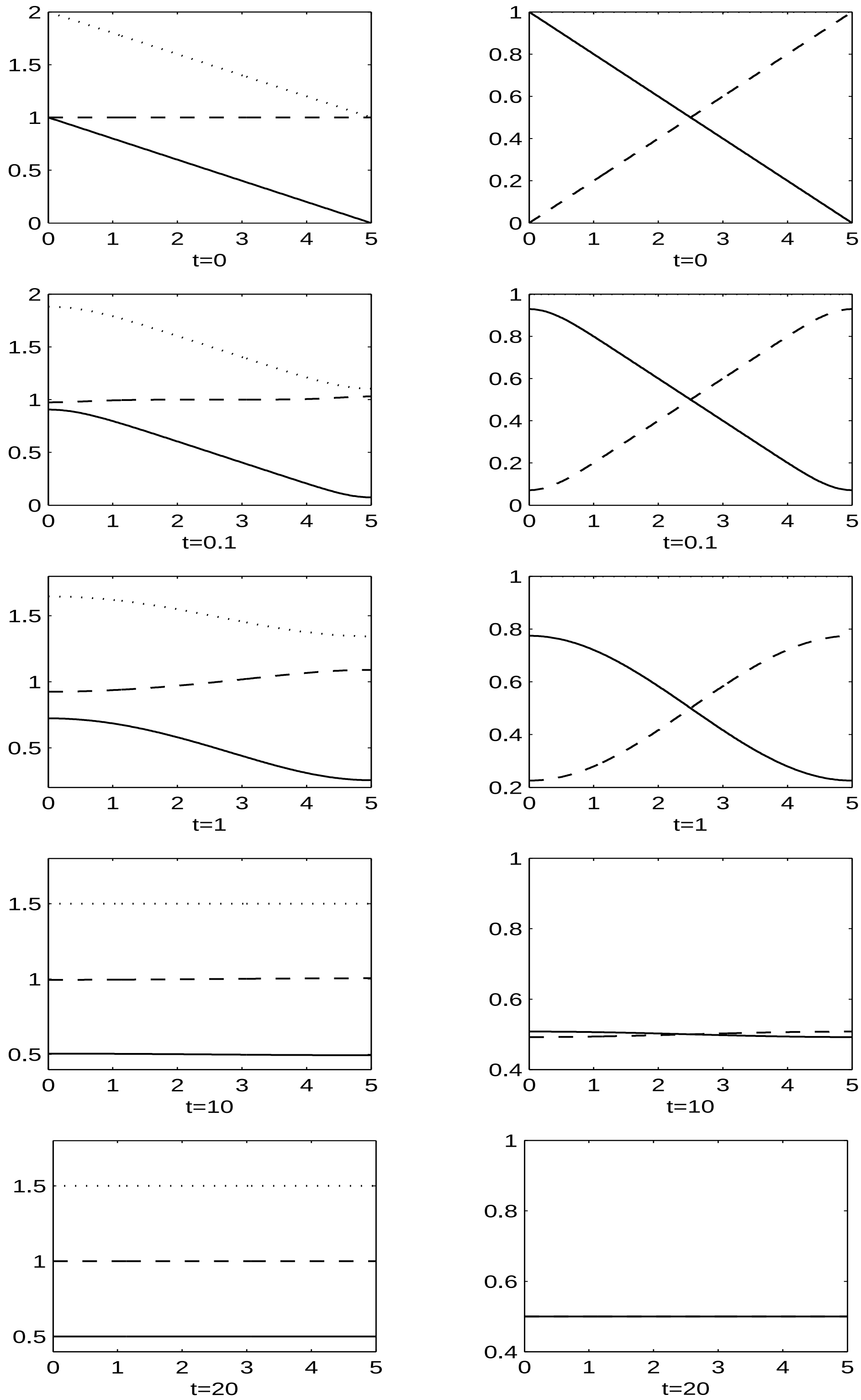
Given $\{U^0_{\varepsilon,1}, U^0_{\varepsilon,2}\} \in S^h \times S^h$, for $k \geq 1$ find $\{U^{n,k}_{\varepsilon,1}, U^{n,k}_{\varepsilon,2}\} \in S^h \times S^h$ such that for $i = 1$ and 2, and for all $\chi \in S^h$

$$\left(\tfrac{U^{n,k}_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}}{\Delta t_n}, \chi\right)^h + \left(D\, \nabla U^{n,k}_{\varepsilon,i} + \Lambda_\varepsilon(U^{n,k-1}_{\varepsilon,i})\, \nabla \left(U^{n,k}_{\varepsilon,1} + U^{n,k}_{\varepsilon,2}\right), \nabla\chi\right)$$
$$= \left(\gamma_i\, U^{n,k}_{\varepsilon,i} - U^{n,k}_{\varepsilon,i}\, \left[\phi_\varepsilon(U^{n-1}_{\varepsilon,1}) + \phi_\varepsilon(U^{n-1}_{\varepsilon,2})\right], \chi\right)^h. \qquad (5.1.3)$$

On noting the relationship between (P) and $(\widetilde{\mathrm{P}}_M)$, it is obvious that the resulting solutions from solving the iterative system (5.1.3) can be considered as approximate solutions of problem (P) if the number M is chosen sufficiently large such that $||U^{n,k}_{\varepsilon,i}||_{0,\infty} \leq M$, $i = 1$ and 2, can be guaranteed for all n and k. This has been experimentally verified by repeating the experiments in Figure 5.1. For $\gamma = 1$, the solution was graphically identical to Figure 5.1(a). This is expected as both $U_{\varepsilon,1}$ and $U_{\varepsilon,2}$ do not exceed $M = 1$. In contrast, for $\gamma = 2$, one has to increase the number M in order to obtain the same solution as in Figure 5.1(b).

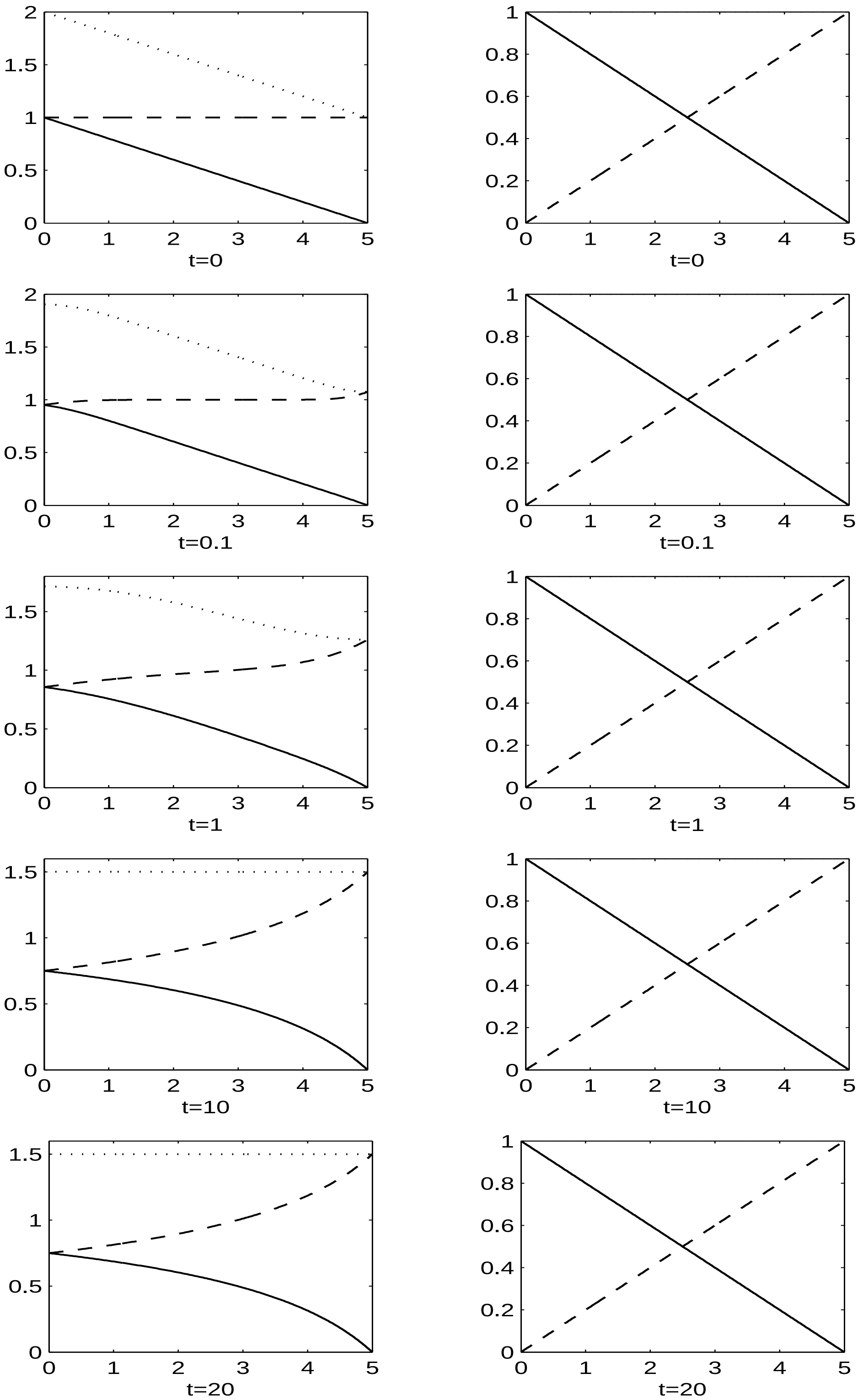
Problem (P_0):

In the last part of this chapter we discuss some numerical results concerning the theoretical aspects in Theorem 4.4.1. For the numerical solutions of problem (P_0), which is (P) with no reaction terms, we use a modified version of the iterative system (5.1.3); (we set the right hand side in (5.1.3) to be zero). In Figure 5.9(a), numerical solutions of (P_0), with $D = 1$, are obtained for the initial data $u^0_1(x) = -0.2\,x + 1$ and $u^0_2(x) = 1$. Each variable converges to the mean integral of its own initial state as t increases. This agrees with what we expect from Theorem 4.4.1. The same behaviour was observed for the initial data $u^0_1(x) = -0.2\,x + 1$ and $u^0_2(x) = 0.2\,x$; see Figure 5.9(b). When we repeated the experiment in Figure 5.9 for $D = 0$, we found that the solutions behaved differently; see Figure 5.10. Finally, we note that the results in Figure 5.10 agree with the experimental findings in [55].



(a) (b)

Figure 5.9: Numerical solutions of $(\widetilde{\mathrm{P}}^{h,\Delta t}_{0,M,\varepsilon})$ for $D = 1$ and $M = 10$. The initial data are: (a) $u_1^0(x) = -0.2\,x + 1$ and $u_2^0(x) = 1$; (b) $u_1^0(x) = -0.2\,x + 1$ and $u_2^0(x) = 0.2\,x$. The solid, dashed and dotted lines represent u_1, u_2 and $u_1 + u_2$, respectively.



(a) (b)

Figure 5.10: Numerical solutions of $(\widetilde{\mathrm{P}}^{h,\Delta t}_{0,M,\varepsilon})$ for $D = 0$ and $M = 10$. The initial data are: (a) $u_1^0(x) = -0.2\,x + 1$ and $u_2^0(x) = 1$; (b) $u_1^0(x) = -0.2\,x + 1$ and $u_2^0(x) = 0.2\,x$. The solid, dashed and dotted lines represent u_1, u_2 and $u_1 + u_2$, respectively.

Chapter 6

The axial segregation model: Analysis and results

6.1 Motivation

In this chapter we present a mathematical study of the problem (Q) introduced in Section 1.2. At first, we mention that the system (1.2.1a)-(1.2.1d) has been considered mathematically in a recent work by Galiano *et al.* [34]. Mainly, existence of a global in-time weak solution of the system has been proved using entropy-type inequalities and approximation arguments. The main difficulty of the analysis is due to the cross diffusion nonlinear term $\nabla \cdot (\,(1-w^2)\,\nabla z\,)$. This is treated by defining a non-negative function $\Phi : (1\,,-1) \rightarrow \mathbb{R}^{\geq 0}$ satisfying

$$\nabla\,[\Phi'(w)] = \frac{\lambda\,\nabla w}{1-w^2} \qquad \text{with} \quad \Phi(0) = 0\,;$$

that is for all $s \in (1\,,-1)$

$$\Phi(s) := \tfrac{\lambda}{2}\,\left[\,(1+s)\,\ln(1+s) + (1-s)\,\ln(1-s)\,\right]. \tag{6.1.1}$$

On noting (6.1.1) and that $\lambda > 0$, it is convenient for later purposes to define the $\frac{2}{\lambda}$-Lipschitz continuous function $\mathcal{V} : [1\,,-1] \rightarrow [0\,,\lambda^{-1}]$ given by

$$\mathcal{V}(s) := [\Phi''(s)]^{-1} = \frac{1-s^2}{\lambda}\,. \tag{6.1.2}$$

Testing (1.2.1a) with $\Phi'(w)$ and (1.2.1b) with z and using the boundary condition (1.2.1c) and (6.1.2) leads us to obtaining the following entropy functional

$$E(t) = \int_\Omega \left(\Phi(w) + \tfrac{1}{2}\, z^2 \right) \mathrm{d}x \ \geq 0,$$

with the corresponding entropy inequality

$$E(t) + \int_0^t \left(\rho\, \lambda\, ||\nabla w||_0^2 + ||z||_1^2 \right) \mathrm{d}t \ \leq E(0) + \mu \int_0^t \int_\Omega w\, z \ \mathrm{d}x \ \mathrm{d}t\,. \tag{6.1.3}$$

Since the values $w = \pm 1$ are possible, we note that the inequality above is not generally valid.

In [34], as it will be in our analysis, the estimate (6.1.3) played a central role to show the existence of a global weak solution to the system (1.2.1a)-(1.2.1d). That was achieved using an exponential transformation with a change of variables to overcome the singular nature at $w = \pm 1$, and utilizing a time semi-discrete approximation and standard compactness arguments to show the existence.

Instead of (1.2.1c), the authors in [34] solved the system (1.2.1a)-(1.2.1b) using the following periodic boundary conditions:

$$\begin{aligned} w(0,\cdot) = w(L,\cdot)\ ,\ \nabla w(0,\cdot) = \nabla w(L,\cdot) \\ z(0,\cdot) = z(L,\cdot)\ ,\ \nabla z(0,\cdot) = \nabla z(L,\cdot) \end{aligned} \qquad \text{in } (0,T). \tag{6.1.4}$$

However, for consistency with the analysis of problem (P), we impose the Neumann-type boundary conditions (1.2.1c). The subsequent work in this chapter is also valid for periodic boundary conditions of the type (6.1.4); see Remark 6.6.3.

Our aim in this chapter is to study the system (1.2.1a)-(1.2.1d) using a finite element method. Namely, we use the framework presented above, for the problem (P), to prove existence of a weak solution of (Q). Furthermore, we discuss some uniqueness results and obtain some error estimates.

As in Section 2.3, we deal with the singularity at $w = \pm 1$ by using a regularization procedure. Then we propose and analyse a fully discrete finite element approximation of (Q). It will be clear that the tools and arguments provided in the previous chapters are significantly contributed to the analysis of the current chapter.

The layout of the rest of the chapter is as follows. In Section 6.2 a regularized problem of (Q) is considered and hence a well defined entropy inequality is established. In Section 6.3 a fully discrete finite element approximation to (Q) is proposed. Additional notation to that presented previously is also included. Existence of fully discrete solutions is shown under an appropriate assumption on the time discretization parameter. A discrete analogue entropy inequality is derived and some stability bounds of the approximations are shown. Finally, the uniqueness of the fully discrete approximations is discussed. In Section 6.4 , the convergence of our approximation is established and hence existence of a global weak solution to the system (1.2.1a)-(1.2.1d) is shown. In the last part of Section 6.4 , the uniqueness of solutions in a slightly smaller class of functions is rediscovered for sufficiently small cross diffusion parameter λ . Section 6.5 is devoted to the discussion of an error bound between the approximations and the weak solutions of (Q). Finally, in Section 6.6 the long time behaviour of the solutions of (Q) is discussed.

6.2 A regularized problem

In order to make the key inequality (6.1.3) well defined, we introduce an alternative approach to the one considered in [34] that relies on a change of variables. Namely, we adapt the regularization procedure that was employed by Elliott and Luckhaus, in [29], to study a Cahn-Hilliard equation.

We replace the function $\Phi(s)$ by the twice continuously differentiable function $\Phi_\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$, where $\varepsilon \in (0, 1)$ and

$$\Phi_\varepsilon(s) := \begin{cases} \frac{\lambda}{2} \left[(1+s) \ln(1+s) + \frac{1}{2\varepsilon} (1-s)^2 + (1-s) \ln \varepsilon - \frac{\varepsilon}{2} \right] & \text{if} \quad s \geq 1-\varepsilon \,, \\ \Phi(s) := \frac{\lambda}{2} \left[(1+s) \ln(1+s) + (1-s) \ln(1-s) \right] & \text{if } |s| \leq 1-\varepsilon \,, \\ \frac{\lambda}{2} \left[(1-s) \ln(1-s) + \frac{1}{2\varepsilon} (1+s)^2 + (1+s) \ln \varepsilon - \frac{\varepsilon}{2} \right] & \text{if} \quad s \leq \varepsilon - 1 \,; \end{cases} \tag{6.2.1a}$$

with an increasing first derivative

$$\Phi'_\varepsilon(s) := \begin{cases} \frac{\lambda}{2}\left[1+\ln(1+s)-\frac{1}{\varepsilon}(1-s)-\ln\varepsilon\right] & \text{if } \ s \geq 1-\varepsilon\,, \\ \Phi'(s) := \frac{\lambda}{2}\left[\ln(1+s)-\ln(1-s)\right] & \text{if } |s| \leq 1-\varepsilon\,, \\ \frac{\lambda}{2}\left[-1-\ln(1-s)+\frac{1}{\varepsilon}(1+s)+\ln\varepsilon\right] & \text{if } \ s \leq \varepsilon-1\,; \end{cases} \tag{6.2.1b}$$

and with a positive second derivative

$$\Phi''_\varepsilon(s) := \begin{cases} \frac{\lambda}{2}\left[\frac{1}{1+s}+\frac{1}{\varepsilon}\right] & \text{if } \ s \geq 1-\varepsilon\,, \\ \Phi''(s) := \frac{\lambda}{1-s^2} & \text{if } |s| \leq 1-\varepsilon\,, \\ \frac{\lambda}{2}\left[\frac{1}{1-s}+\frac{1}{\varepsilon}\right] & \text{if } \ s \leq \varepsilon-1\,. \end{cases} \tag{6.2.1c}$$

We also define the $\frac{2}{\lambda}$-Lipschitz continuous function $\mathcal{V}_\varepsilon : \mathbb{R} \to \mathbb{R}^{>0}$ given by

$$\mathcal{V}_\varepsilon(s) := [\Phi''_\varepsilon(s)]^{-1} := \begin{cases} \frac{2}{\lambda}\left[\frac{\varepsilon\,(1+s)}{\varepsilon+1+s}\right] & \text{if } \ s \geq 1-\varepsilon\,, \\ \mathcal{V}(s) := \frac{1-s^2}{\lambda} & \text{if } |s| \leq 1-\varepsilon\,, \\ \frac{2}{\lambda}\left[\frac{\varepsilon\,(1-s)}{\varepsilon+1-s}\right] & \text{if } \ s \leq \varepsilon-1\,. \end{cases} \tag{6.2.2}$$

For later use, we note that the regularized functions $\Phi_\varepsilon(s)$ and $\mathcal{V}_\varepsilon(s)$ have the following easily established properties:

- For all $\varepsilon \in (0,1)$

$$\Phi_\varepsilon(s) \leq \lambda \ln 2 \qquad \forall\ |s| \leq 1\,, \tag{6.2.3a}$$

$$\Phi_\varepsilon(s) \geq \tfrac{\lambda}{4}\, s^2 - \tfrac{\lambda}{2} \qquad \forall\ s \in \mathbb{R}\,. \tag{6.2.3b}$$

- For all $\varepsilon \in (0, \frac{1}{2}\,]$ and for all $r\,,\ s \in \mathbb{R}$

$$\begin{aligned}(r-s)\,\Phi'_\varepsilon(r) &\geq \Phi_\varepsilon(r) - \Phi_\varepsilon(s) + \tfrac{\lambda}{2}\,(r-s)^2 \\ &\geq \Phi_\varepsilon(r) - \Phi_\varepsilon(s) + \tfrac{\lambda}{4}\, r^2 - \tfrac{\lambda}{2}\, s^2\,. \end{aligned} \tag{6.2.4}$$

- For all $\varepsilon \in (0, \frac{1}{2}\,]$ and for all $s \in \mathbb{R}$

$$\tfrac{\varepsilon}{\lambda} \leq \tfrac{\varepsilon\,(2-\varepsilon)}{\lambda} \leq \mathcal{V}_\varepsilon(s) \leq \tfrac{1}{\lambda}\,. \tag{6.2.5}$$

- For all $\varepsilon \in (0,1)$ and for all $|s| \leq 1$

$$0 \leq \mathcal{V}_\varepsilon(s) - \mathcal{V}(s) \leq \tfrac{2\varepsilon}{\lambda}\,. \tag{6.2.6}$$

In addition, the function $\Phi_\varepsilon(s)$ has the following key property which we will require to show that the derived solution w belongs to $[1\,,-1]$.

- For all $\varepsilon \in (0,1)$ and for all $s \in \mathbb{R}$

$$\Phi_\varepsilon(s) \geq \tfrac{\lambda}{4\varepsilon}\,\big([s-1]_+^2 + [-1-s]_+^2\big) - \tfrac{\lambda}{4}\,. \tag{6.2.7}$$

To see (6.2.7), we firstly note for $|s| \leq 1$ that

$$\Phi_\varepsilon(s) \geq \Phi_\varepsilon(0) = 0 > -\tfrac{\lambda}{4}\,.$$

Secondly, for the case $s \geq 1$ we have that

$$\Phi_\varepsilon(s) \geq \tfrac{\lambda}{2}\,\big(\tfrac{1}{2\varepsilon}\,(1-s)^2 - \tfrac{\varepsilon}{2}\big) \geq \tfrac{\lambda}{4\varepsilon}\,\big([s-1]_+^2 + [-1-s]_+^2\big) - \tfrac{\lambda}{4}\,.$$

Finally, similarly to the case $s \geq 1$, the inequality (6.2.7) holds for all $s \leq -1$.

We now introduce for $\varepsilon \in (0, \frac{1}{2}]$ the corresponding regularized version of the problem (Q):

($\mathbf{Q}_\varepsilon$) Find $\{w_\varepsilon(x,t), z_\varepsilon(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\frac{\partial w_\varepsilon}{\partial t} = \nabla \cdot (\,\rho\, \nabla w_\varepsilon - \lambda\, \mathcal{V}_\varepsilon(w_\varepsilon)\, \nabla z_\varepsilon\,) \qquad \text{in } \Omega_T, \tag{6.2.8a}$$

$$\frac{\partial z_\varepsilon}{\partial t} = \nabla \cdot (\,\nabla z_\varepsilon + \lambda\, \nabla w_\varepsilon\,) + \mu\, w_\varepsilon - z_\varepsilon \qquad \text{in } \Omega_T, \tag{6.2.8b}$$

with boundary conditions

$$\left.\begin{aligned}
&[\rho\, \nabla w_\varepsilon - \lambda\, \mathcal{V}_\varepsilon(w_\varepsilon)\, \nabla z_\varepsilon]\,(0,\cdot) = [\nabla z_\varepsilon + \lambda\, \nabla w_\varepsilon]\,(0,\cdot) = 0,\\
&[\rho\, \nabla w_\varepsilon - \lambda\, \mathcal{V}_\varepsilon(w_\varepsilon)\, \nabla z_\varepsilon]\,(L,\cdot) = [\nabla z_\varepsilon + \lambda\, \nabla w_\varepsilon]\,(L,\cdot) = 0,
\end{aligned}\right\} \quad \text{in } (0,T), \tag{6.2.8c}$$

and initial conditions

$$w_\varepsilon(x,0) = w^0(x)\ ,\ z_\varepsilon(x,0) = z^0(x) \qquad \forall\, x \in \Omega\,. \tag{6.2.8d}$$

In the following lemma we establish a well defined entropy inequality to the system (6.2.8a)-(6.2.8d) which will play a central role in the numerical analysis that follows.

Lemma 6.2.1 Let $\rho\,,\ \lambda > 0$ and $\mu \geq 0$ and let $w^0,\ z^0\ \in L^2(\Omega)$ with $|w^0(\cdot)| \leq 1$ *a.e.* in Ω. Then there exists a positive $C(w^0,\ z^0\,,\ \rho\,,\ \lambda\,,\ \mu\,,\ C_p)$ independent of ε such that any solution $\{w_\varepsilon, z_\varepsilon\}$ of (Q_ε) satisfies

$$\sup_{0<t<T} \int_\Omega \left(\Phi_\varepsilon(w_\varepsilon) + \tfrac{1}{2}\, z_\varepsilon^2 \right)\, \mathrm{d}x \ + \int_0^T \left(\|\nabla w_\varepsilon\|_1^2 + \|z_\varepsilon\|_1^2 \right)\, \mathrm{d}t \ \leq C\,. \tag{6.2.9}$$

In addition,

$$\sup_{0<t<T} \int_\Omega \left([w_\varepsilon - 1]_+^2 + [-1 - w_\varepsilon]_+^2 \right) \mathrm{d}x \leq C\,\varepsilon. \tag{6.2.10}$$

Proof: Testing (6.2.8a) with $\Phi_\varepsilon'(w_\varepsilon)$ and (6.2.8b) with z_ε and summing the resulting equations yields, after using (6.2.8c), that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\Phi_\varepsilon(w_\varepsilon) + \tfrac{1}{2}\, z_\varepsilon^2 \right)\ \mathrm{d}x \ + \rho \int_\Omega \Phi_\varepsilon''(w_\varepsilon)\, |\nabla w_\varepsilon|^2\ \mathrm{d}x \\ + \int_\Omega \left(|\nabla z_\varepsilon|^2 + |z_\varepsilon|^2 \right)\ \mathrm{d}x = \mu \int_\Omega w_\varepsilon\, z_\varepsilon\ \mathrm{d}x\,,\end{aligned} \tag{6.2.11}$$

where we have noticed from (6.2.1c) and (6.2.2) that

$$\mathcal{V}_\varepsilon(w_\varepsilon) \nabla \left[\Phi_\varepsilon'(w_\varepsilon) \right] = \nabla w_\varepsilon\,. \tag{6.2.12}$$

It follows immediately from (6.2.8a) and (6.2.8d) for *a.e.* $t \in (0, T)$ that

$$(w_\varepsilon(\cdot, t), 1) = (w_\varepsilon(\cdot, 0), 1) = (w^0(\cdot), 1)\,. \tag{6.2.13}$$

We now obtain from the Young's inequality, the Poincaré inequality and (6.2.13), for positive constant μ, that

$$\mu \int_\Omega w_\varepsilon\, z_\varepsilon\ \mathrm{d}x \ \leq\ \tfrac{\rho\lambda}{2\,C_p}\, \|w_\varepsilon\|_0^2 + C\, \|z_\varepsilon\|_0^2 \ \leq\ \tfrac{\rho\lambda}{2}\, |w_\varepsilon|_1^2 + C\, \left[\, 1 + \|z_\varepsilon\|_0^2 \,\right]. \tag{6.2.14}$$

Combining (6.2.11), (6.2.14), (6.2.5) and noting that $\Phi_\varepsilon(s) \geq 0$ leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\Phi_\varepsilon(w_\varepsilon) + \tfrac{1}{2}\, z_\varepsilon^2 \right)\ \mathrm{d}x \ + \tfrac{\rho\lambda}{2}\, |w_\varepsilon|_1^2 + \|z_\varepsilon\|_1^2 \\ \leq C \left(1 + \int_\Omega \left(\Phi_\varepsilon(w_\varepsilon) + \tfrac{1}{2}\, z_\varepsilon^2 \right)\ \mathrm{d}x \right).\end{aligned} \tag{6.2.15}$$

Hence, on noting the assumptions on the initial data and (6.2.3a), the result (6.2.9) follows from (6.2.15) after a simple application of the Grönwall lemma. Finally, the result (6.2.10) follows immediately from the first bound in (6.2.9) and (6.2.7). $\square$

Obviously, the regularized entropy inequality (6.2.9) and the estimate (6.2.10) can be used to pass to the limit $\varepsilon \to 0$ in (Q_ε) in order to obtain existence of a solution to (Q). In the following section we formulate and analyse a fully discrete finite element approximation of the regularized system (6.2.8a)-(6.2.8d).

6.3 A fully discrete approximation

6.3.1 An approximation problem

Let $0 = x_0 < x_1 < \ldots < x_{J-1} < x_J = L$ be a partitioning of the domain $\Omega := (0, L)$ into the open simplices $\kappa_j := (x_{j-1}, x_j)$, $j = 1, \cdots, J$, with $h_j := x_j - x_{j-1}$ and $h := \max_{j=1,\cdots,J} h_j$. In addition, let $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$ be a partitioning of $(0, T)$ into time steps $\Delta t_n := t_n - t_{n-1}$, $n = 1, \cdots, N$, with $\Delta t := \max_{n=1,\cdots,N} \Delta t_n$. For any $\varepsilon \in (0, \frac{1}{2}]$ we define the piecewise constant function $\Pi_\varepsilon : S^h \to L^\infty(\Omega)$ such that for $j = 1, \cdots, J$

$$\Pi_\varepsilon(\chi)\,|_{\kappa_j} := \begin{cases} \frac{\chi(x_j)-\chi(x_{j-1})}{\Phi'_\varepsilon(\chi(x_j))-\Phi'_\varepsilon(\chi(x_{j-1}))} = \frac{1}{\Phi''_\varepsilon(\chi(\zeta))} & \\ \qquad \text{for some } \zeta \in \kappa_j & \text{if } \chi(x_j) \neq \chi(x_{j-1}), \\ \frac{1}{\Phi''_\varepsilon(\chi(x_j))} & \text{if } \chi(x_j) = \chi(x_{j-1}). \end{cases} \tag{6.3.1}$$

Clearly, the function Π_ε satisfies for all $\chi \in S^h$ and for *a.e.* in Ω the discrete analogue of (6.2.12)

$$\Pi_\varepsilon(\chi)\, \nabla \pi^h[\Phi'_\varepsilon(\chi)] = \nabla \chi\,. \tag{6.3.2}$$

Lemma 6.3.1 For any $\varepsilon \in (0, \frac{1}{2}]$, the function $\Pi_\varepsilon : S^h \to L^\infty(\Omega)$ satisfies *a.e.* in Ω that

$$\tfrac{\varepsilon}{\lambda} \leq \Pi_\varepsilon(\chi) \leq \tfrac{1}{\lambda} \qquad \forall\, \chi \in S^h. \tag{6.3.3}$$

In addition, it holds for all $\chi_1\,,\ \chi_2 \in S^h$ and for $j = 1, \cdots, J$ that

$$\begin{aligned}
&\big|(\Pi_\varepsilon(\chi_1) - \Pi_\varepsilon(\chi_2))\,|_{\kappa_j}\big| \\
&\qquad \le \tfrac{2}{\lambda}\, \max_{s\in\mathbb{R}} [\Phi''_\varepsilon(s)]\, \max_{s\in\mathbb{R}} [\mathcal{V}_\varepsilon(s)]\, [\,|\chi_1(x_j) - \chi_2(x_j)| + |\chi_1(x_{j-1}) - \chi_2(x_{j-1})|\,] \\
&\qquad \le \tfrac{2}{\lambda\,\varepsilon}\, [\,|\chi_1(x_j) - \chi_2(x_j)| + |\chi_1(x_{j-1}) - \chi_2(x_{j-1})|\,] \\
&\qquad \le \tfrac{4}{\lambda\,\varepsilon}\, \|\chi_1 - \chi_2\|_{0,\infty}\,. \qquad (6.3.4)
\end{aligned}$$

Proof: The bound (6.3.3) follows immediately from (6.3.1), (6.2.2) and (6.2.5). The proof of (6.3.4) is a simple modification of the proof of Lemma 2.4.4 where we recall that $\mathcal{V}_\varepsilon(s)$ is $\frac{2}{\lambda}$-Lipschitz continuous function. □

Lemma 6.3.2 For any given $\varepsilon \in (0, \frac{1}{2}\,]$ the function $\Pi_\varepsilon : S^h \to L^\infty(\Omega)$ is such that for $j = 1, \cdots, J$

$$\max_{x\in\kappa_j} |\Pi_\varepsilon(\chi(x)) - \mathcal{V}_\varepsilon(\chi(x))| \le \tfrac{2}{\lambda}\, h_j\, |\nabla\chi\,|_{\kappa_j}\,| \qquad \forall\ \chi \in S^h. \qquad (6.3.5)$$

Proof: It follows easily from (6.3.1), (6.2.2) and the Lipschitz continuity of $\mathcal{V}_\varepsilon$. □

Now, we propose the following fully discrete finite element approximation of (Q_ε) for any $\varepsilon \in (0, \frac{1}{2}\,]$:

$(\mathbf{Q}_\varepsilon^{h,\Delta t})$ For $n \ge 1$ find $\{W_\varepsilon^n, Z_\varepsilon^n\} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$\left(\tfrac{W_\varepsilon^n - W_\varepsilon^{n-1}}{\Delta t_n}, \chi\right)^h + \rho\, (\nabla W_\varepsilon^n, \nabla\chi) - \lambda\, (\Pi_\varepsilon(W_\varepsilon^n)\, \nabla Z_\varepsilon^n, \nabla\chi) = 0\,, \qquad (6.3.6a)$$

$$\begin{aligned}
&\left(\tfrac{Z_\varepsilon^n - Z_\varepsilon^{n-1}}{\Delta t_n}, \chi\right)^h + (Z_\varepsilon^n, \chi)^h + (\nabla Z_\varepsilon^n, \nabla\chi) + \lambda\, (\nabla W_\varepsilon^n, \nabla\chi) \\
&\qquad\qquad = \mu\, \left(\theta\, W_\varepsilon^n + (1-\theta)\, W_\varepsilon^{n-1}, \chi\right)^h, \qquad (6.3.6b)
\end{aligned}$$

where $\theta \in [0, 1]$, and $W_\varepsilon^0 \in S^h$ and $Z_\varepsilon^0 \in S^h$ are given approximations of w^0 and z^0 respectively.

In addition to the operator $\mathcal{G}$ introduced in Section 3.1, our analysis of the system (6.3.6a)-(6.3.6b) will require us to use the operator $\widetilde{\mathcal{G}} : \mathcal{F} \to \mathcal{K}$ given by

$$(\nabla\widetilde{\mathcal{G}}v, \nabla\eta) = \langle v, \eta\rangle \qquad \forall\ \eta \in H^1(\Omega), \qquad (6.3.7)$$

where

$$\mathcal{F} = \{ v \in (H^1(\Omega))' \ : \ \langle v, 1 \rangle = 0 \} ,$$
$$\mathcal{K} = \{ \eta \in H^1(\Omega) \ : \ (\eta, 1) = 0 \} .$$

The existence and the uniqueness of $\widetilde{\mathcal{G}}v$, for a given $v \in \mathcal{F}$, follows from the Lax-Milgram theorem, see Appendix A.1.2, and the Poincaré inequality. We now define the following norm on the set $\mathcal{F}$

$$\|v\|_{-1} := |\widetilde{\mathcal{G}}v|_1 = \langle v, \widetilde{\mathcal{G}}v \rangle^{\frac{1}{2}} \qquad \forall\ v \in \mathcal{F} . \tag{6.3.8}$$

Using the definition of the dual norm, the Cauchy-Schwarz inequality and the Poincaré inequality, one can easily obtain from (6.3.7) and (6.3.8) that

$$\|v\|_{(H^1(\Omega))'} \leq \|v\|_{-1} \leq C\, \|v\|_{(H^1(\Omega))'}. \tag{6.3.9}$$

For all $v \in L^2(\Omega) \cap \mathcal{F}$ and for all $\delta > 0$ we have from (6.3.7) and the Young's inequality that

$$(v, \eta) = (\nabla \widetilde{\mathcal{G}}v, \nabla \eta) \leq |\widetilde{\mathcal{G}}v|_1\, |\eta|_1 \leq \tfrac{\delta}{2}\, |\widetilde{\mathcal{G}}v|_1^2 + \tfrac{1}{2\,\delta}\, |\eta|_1^2 \qquad \forall\ \eta \in H^1(\Omega). \tag{6.3.10}$$

From (6.3.10) and (2.4.14) we have, for appropriate choice of δ, that

$$\|\chi\|_0 \leq C\, h^{-1} |\widetilde{\mathcal{G}}\chi|_1 \qquad \forall\ \chi \in \mathcal{F}^h, \tag{6.3.11}$$

where

$$\mathcal{F}^h = \{ \chi \in S^h \ : \ (\chi, 1) = 0 \} .$$

6.3.2 Existence of approximations

Theorem 6.3.3 Let $\rho\,,\ \lambda > 0\,,\ \mu \geq 0$ and $\theta \in [0, 1]$. Let $\{W_\varepsilon^{n-1},\ Z_\varepsilon^{n-1}\} \in S^h \times S^h$ is given for some $n = 1, \cdots, N$. Then for all $\varepsilon \in (0, \frac{1}{2}]$, for all $h > 0$ and for all $\Delta t_n > 0$ such that $\Delta t_n < \frac{2\,\lambda}{\theta\,\mu^2}$ if $\mu \neq 0$ and $\theta \neq 0$, there exists a solution $\{W_\varepsilon^n, Z_\varepsilon^n\} \in S^h \times S^h$ to the n-th step of $(\mathrm{Q}_\varepsilon^{h,\Delta t})$.

Proof: We adapt the argument employed in Theorem 2.4.7. First we define $A_w : S^h \times S^h \to S^h$ and $A_z : S^h \times S^h \to S^h$ such that for all $\chi \in S^h$

$$(A_w(W,Z),\chi)^h = \left(W - W_\varepsilon^{n-1},\chi\right)^h + \rho\,\Delta t_n\;(\nabla W,\nabla\chi) - \lambda\,\Delta t_n\;(\Pi_\varepsilon(W)\,\nabla Z,\nabla\chi)\,, \tag{6.3.12a}$$

$$\begin{aligned}(A_z(W,Z),\chi)^h = {} & \left(Z - Z_\varepsilon^{n-1},\chi\right)^h + \Delta t_n\;(Z,\chi)^h + \Delta t_n\;(\nabla Z,\nabla\chi) \\ & + \lambda\,\Delta t_n\;(\nabla W,\nabla\chi) - \mu\,\Delta t_n\;\left(\theta\,W + (1-\theta)\,W_\varepsilon^{n-1},\chi\right)^h, \end{aligned} \tag{6.3.12b}$$

respectively. It is simple matter to show that the functions A_w and A_z are well defined. Furthermore, for any $R > 0$ it can be easily shown using (6.3.12a)-(6.3.12b), (6.3.3) and (6.3.4) that the functions A_w and A_z are both continuous on the convex compact subset $[S^h]^2_R$; see the proof of Lemma 2.4.6.

It is clear that solving the system (6.3.6a)-(6.3.6b) is equivalent to finding $\{W,Z\} \in S^h \times S^h$ such that

$$A_w(W,Z) = 0 \quad \text{and} \quad A_z(W,Z) = 0\,.$$

By contradiction, let $R > 0$ and assume that there does not exist $\{W,Z\} \in [S^h]^2_R$ with $A_w(W,Z) = A_z(W,Z) = 0$. Hence, on noting the continuity of the functions A_w and A_z on $[S^h]^2_R$, we define the continuous function $B : [S^h]^2_R \to [S^h]^2_R$ given by

$$B(W,Z) := (B_w(W,Z), B_z(W,Z))$$

where

$$\begin{aligned} B_w(W,Z) &:= \frac{-R\,A_w(W,Z)}{|(A_w(W,Z),A_z(W,Z))|_{S^h\times S^h}}\,, \\ B_z(W,Z) &:= \frac{-R\,A_z(W,Z)}{|(A_w(W,Z),A_z(W,Z))|_{S^h\times S^h}}\,. \end{aligned} \tag{6.3.13}$$

We deduce from the Schauder's theorem, see Appendix A.1.1, that there exists $\{W,Z\} \in [S^h]^2_R$ fixed point of B such that

$$|W|_h^2 + |Z|_h^2 = |B_w(W,Z)|_h^2 + |B_z(W,Z)|_h^2 = R^2\,. \tag{6.3.14}$$

To prove a contradiction for R sufficiently large, we choose $\chi \equiv \pi^h[\Phi'_\varepsilon(W)]$ in (6.3.12a) and $\chi \equiv Z$ in (6.3.12b) yielding on noting (2.4.3) and (6.3.2) that

$$(A_w(W,Z),\Phi'_\varepsilon(W))^h = \left(W - W^{n-1}_\varepsilon, \Phi'_\varepsilon(W)\right)^h - \lambda\,\Delta t_n\,(\nabla Z, \nabla W) \\ + \rho\,\Delta t_n\,\left([\Pi_\varepsilon(W)]^{-1}\,\nabla W, \nabla W\right), \quad (6.3.15a)$$

$$(A_z(W,Z),Z)^h = \left(Z - Z^{n-1}_\varepsilon, Z\right)^h + \Delta t_n\,(Z,Z)^h + \Delta t_n\,(\nabla Z,\nabla Z) \\ + \lambda\,\Delta t_n\,(\nabla W, \nabla Z) - \mu\,\Delta t_n\,\left(\theta\,W + (1-\theta)\,W^{n-1}_\varepsilon, Z\right)^h. \quad (6.3.15b)$$

We have from (6.2.4) that

$$(W - W^{n-1}_\varepsilon, \Phi'_\varepsilon(W))^h \geq (\Phi_\varepsilon(W) - \Phi_\varepsilon(W^{n-1}_\varepsilon),1)^h + \tfrac{\lambda}{2}\,|W - W^{n-1}_\varepsilon|^2_h \\ \geq (\Phi_\varepsilon(W) - \Phi_\varepsilon(W^{n-1}_\varepsilon),1)^h + \tfrac{\lambda}{4}\,|W|^2_h - \tfrac{\lambda}{2}\,|W^{n-1}_\varepsilon|^2_h\,. \quad (6.3.16)$$

Using the simple identity

$$2\,s\,(s-r) = s^2 - r^2 + (s-r)^2 \qquad \forall\, r\,,\, s \in \mathbb{R}\,,$$

we obtain that

$$\left(Z - Z^{n-1}_\varepsilon, Z\right)^h \geq \tfrac{1}{2}\,|Z|^2_h - \tfrac{1}{2}\,|Z^{n-1}_\varepsilon|^2_h\,. \quad (6.3.17)$$

It follows from the Young's inequality that

$$\mu\,\Delta t_n\,\left(\theta\,W + (1-\theta)\,W^{n-1}_\varepsilon, Z\right)^h \leq \tfrac{\mu^2\Delta t_n}{4}\left(\theta\,|W|^2_h + (1-\theta)\,|W^{n-1}_\varepsilon|^2_h\right) + \Delta t_n\,|Z|^2_h\,. \quad (6.3.18)$$

Combining (6.3.15a,b), (6.3.16)→(6.3.18), (6.2.3b), (6.3.3) and (6.3.14) and noting the stated assumption on Δt_n yields for R sufficiently large that

$$(A_w(W,Z),\Phi'_\varepsilon(W))^h + (A_z(W,Z),Z)^h \\ \geq \tfrac{1}{2}\left(\lambda - \tfrac{\theta\,\mu^2\Delta t_n}{2}\right)\,|W|^2_h + \tfrac{1}{2}\,|Z|^2_h - C(W^{n-1}_\varepsilon, Z^{n-1}_\varepsilon) \\ \geq \tfrac{1}{2}\,R^2\min\{\lambda - \tfrac{\theta\,\mu^2\Delta t_n}{2}, 1\} - C(W^{n-1}_\varepsilon, Z^{n-1}_\varepsilon) > 0\,. \quad (6.3.19)$$

Further, for R sufficiently large, we have from (6.3.13) and (6.3.19), since $\{W,Z\}$ is fixed point of B, that

$$(W,\Phi'_\varepsilon(W))^h + (Z,Z)^h = (B_w(W,Z),\Phi'_\varepsilon(W))^h + (B_z(W,Z),Z)^h \\ = \frac{-R\,\left[(A_w(W,Z),\Phi'_\varepsilon(W))^h + (A_z(W,Z),Z)^h\right]}{|(A_w(W,Z),A_z(W,Z))|_{S^h\times S^h}} < 0\,. \quad (6.3.20)$$

On the other hand, we have from (6.2.4) and the non-negativity of Φ_ε that

$$(W, \Phi_\varepsilon'(W))^h + (Z, Z)^h \geq (\Phi_\varepsilon(W) - \Phi_\varepsilon(0), 1)^h + \tfrac{\lambda}{2}\, |W|_h^2 + |Z|_h^2 > 0\,,$$

which contradicts (6.3.20). As a result, we conclude that there exists $\{W_\varepsilon^n, Z_\varepsilon^n\} \in S^h \times S^h$ satisfies $A_w(W_\varepsilon^n, Z_\varepsilon^n) = A_z(W_\varepsilon^n, Z_\varepsilon^n) = 0$. Thus, we have existence of a solution to the n-th step of $(\mathrm{Q}_\varepsilon^{h,\Delta t})$. □

6.3.3 Discrete entropy inequality and stability bounds

In the following lemma we obtain a discrete analogue of the estimate (6.2.15):

Lemma 6.3.4 Let the assumptions of Theorem 6.3.3 hold and let $\{W_\varepsilon^{n-1},\, Z_\varepsilon^{n-1}\} \in S^h \times S^h$, $n \geq 1$. Then a solution $\{W_\varepsilon^n, Z_\varepsilon^n\} \in S^h \times S^h$ to the n-th step of $(\mathrm{Q}_\varepsilon^{h,\Delta t})$ satisfies

$$\begin{aligned}(\Phi_\varepsilon(W_\varepsilon^n), 1)^h &+ \left(\tfrac{1}{2} - (\tfrac{\mu^2}{2\,r} - 1)\Delta t_n\right)\, |Z_\varepsilon^n|_h^2 + \rho\,\lambda\,\left(1 - \tfrac{\theta}{2}\right)\,\Delta t_n\, |W_\varepsilon^n|_1^2 + \Delta t_n\, |Z_\varepsilon^n|_1^2 \\ &\leq (\Phi_\varepsilon(W_\varepsilon^{n-1}), 1)^h + \tfrac{1}{2}\, |Z_\varepsilon^{n-1}|_h^2 + \tfrac{r}{2}\,(1-\theta)\,\Delta t_n\, |W_\varepsilon^{n-1}|_h^2 + C\,\Delta t_n\, |(W_\varepsilon^0, 1)|^2,\end{aligned} \tag{6.3.21}$$

where $r = \frac{\rho\,\lambda}{3\,C_p}$, C_p is the positive constant generated from applying the Poincaré inequality (2.1.8).

Proof: Choosing $\chi \equiv \Delta t_n\, \pi^h[\Phi_\varepsilon'(W_\varepsilon^n)]$ in (6.3.6a) and $\chi \equiv \Delta t_n\, Z_\varepsilon^n$ in (6.3.6b) yields on noting (6.3.2), (6.3.3), (6.3.16) and (6.3.17) that

$$(\Phi_\varepsilon(W_\varepsilon^n), 1)^h + \rho\,\lambda\,\Delta t_n\, |W_\varepsilon^n|_1^2 - \lambda\,\Delta t_n\,(\nabla Z_\varepsilon^n, \nabla W_\varepsilon^n) \leq (\Phi_\varepsilon(W_\varepsilon^{n-1}), 1)^h, \tag{6.3.22a}$$

$$\begin{aligned}\tfrac{1}{2}\, |Z_\varepsilon^n|_h^2 &+ \Delta t_n\, |Z_\varepsilon^n|_h^2 + \Delta t_n\, |Z_\varepsilon^n|_1^2 + \lambda\,\Delta t_n\,(\nabla W_\varepsilon^n, \nabla Z_\varepsilon^n) \\ &\leq \tfrac{1}{2}\, |Z_\varepsilon^{n-1}|_h^2 + \mu\,\Delta t_n\,\left(\theta\, W_\varepsilon^n + (1-\theta)\, W_\varepsilon^{n-1}, Z_\varepsilon\right)^h.\end{aligned} \tag{6.3.22b}$$

We also note that testing (6.3.6a) with $\chi \equiv 1$ gives

$$(W_\varepsilon^n, 1) = (W_\varepsilon^0, 1) \qquad n = 1, \cdots, N. \tag{6.3.23}$$

It follows from the Young's inequality, (2.4.2), the Poincaré inequality, (2.1.8), and (6.3.23) that

$$\begin{aligned}
&\mu\,\Delta t_n\,\left(\theta\,W_\varepsilon^n + (1-\theta)\,W_\varepsilon^{n-1}, Z_\varepsilon\right)^h \\
&\qquad\leq \tfrac{r\,\theta}{2}\,\Delta t_n\,|W_\varepsilon^n|_h^2 + \tfrac{r}{2}\,(1-\theta)\,\Delta t_n\,|W_\varepsilon^{n-1}|_h^2 + \tfrac{\mu^2}{2\,r}\Delta t_n\,|Z_\varepsilon^n|_h^2 \\
&\qquad\leq \tfrac{3\,r\,\theta\,C_p}{2}\,\Delta t_n\,|W_\varepsilon^n|_1^2 + \tfrac{r}{2}\,(1-\theta)\,\Delta t_n\,|W_\varepsilon^{n-1}|_h^2 + \tfrac{\mu^2}{2\,r}\Delta t_n\,|Z_\varepsilon^n|_h^2 + C\,\Delta t_n\,|(W_\varepsilon^0, 1)|^2.
\end{aligned} \tag{6.3.24}$$

Under the stated choice of r, we obtain the desired result (6.3.21) by adding (6.3.22a,b) and noting (6.3.24). $\square$

Lemma 6.3.5 Let $w^0,\ z^0\ \in L^2(\Omega)$ with $|w^0(\cdot)| \leq 1$ *a.e.* in Ω. Further, let either $W_\varepsilon^0 \equiv P^h w^0\,,\ Z_\varepsilon^0 \equiv P^h z^0$; or $W_\varepsilon^0 \equiv \pi^h w^0\,,\ Z_\varepsilon^0 \equiv \pi^h z^0$ if $w^0\,,\ z^0 \in C(\overline{\Omega})$. Then there exists a positive C independent of h, Δt and ε such that

$$\|W_\varepsilon^0\|_0 + \|Z_\varepsilon^0\|_0 + (W_\varepsilon^0, 1) + (\Phi_\varepsilon(W_\varepsilon^0), 1)^h \ \leq C\,. \tag{6.3.25}$$

Moreover, it holds that

$$|W_\varepsilon^0| \leq 1 \qquad \text{in } \Omega. \tag{6.3.26}$$

Proof: The first three bounds in (6.3.25) and (6.3.26) follow immediately from (3.1.1), the definition of the interpolation operator π^h and (3.1.2) on recalling our assumptions on the initial data. The last bound in (6.3.25) follows from (2.4.1) and (6.2.3a) on noting (6.3.26). $\square$

In the following theorem we derive a discrete entropy inequality of the system (6.3.6a)-(6.3.6b) that is consistent with the entropy inequality obtained in Lemma 6.2.1 .

Theorem 6.3.6 Let $\rho\,,\ \lambda > 0\,,\ \mu \geq 0$ and $\theta \in [0,1]$ and let $w^0,\ z^0\ \in L^2(\Omega)$ with $|w^0(\cdot)| \leq 1$ *a.e.* in Ω. Let either $W_\varepsilon^0 \equiv P^h w^0\,,\ Z_\varepsilon^0 \equiv P^h z^0$; or $W_\varepsilon^0 \equiv \pi^h w^0\,,\ Z_\varepsilon^0 \equiv \pi^h z^0$ if $w^0\,,\ z^0 \in C(\overline{\Omega})$. Further, let $\varepsilon \in (0, \frac{1}{2}]$, $h > 0$ and $\Delta t > 0$ be such that

(i) $\Delta t < \frac{2\,\lambda}{\theta\,\mu^2}$ if $\mu \neq 0$ and $\theta \neq 0\,$;
(ii) $\left(\frac{\mu^2}{2\,r} - 1\right)\Delta t \leq \frac{1}{2} - \delta$ for some $\delta \in (0, \frac{1}{2})$ if $r < \frac{\mu^2}{2}$ where $r = \frac{\rho\,\lambda}{3\,C_p}\,$;
(iii) $\Delta t_n \leq \Delta t_{n-1} \quad \forall\ n = 2, \cdots, N$.

Then a solution $\{W_\varepsilon^n, Z_\varepsilon^n\}_{n=1}^N$ to $(\mathrm{Q}_\varepsilon^{h,\Delta t})$ is such that

$$\max_{n=1,\cdots,N} \left[(\Phi_\varepsilon(W_\varepsilon^n),1)^h + \|W_\varepsilon^n\|_0^2 + \|Z_\varepsilon^n\|_0^2 \right] + \sum_{n=1}^N \Delta t_n \left[\|W_\varepsilon^n\|_1^2 + \|Z_\varepsilon^n\|_1^2 \right] \leq C\,. \tag{6.3.27}$$

In addition,

$$\max_{n=1,\cdots,N} \left[\|\pi^h[W_\varepsilon^n - 1]_+\|_0^2 + \|\pi^h[-1 - W_\varepsilon^n]_+\|_0^2 \right] \leq C\,\varepsilon\,. \tag{6.3.28}$$

Furthermore,

$$\sum_{n=1}^N \Delta t_n \left[\|\tfrac{W_\varepsilon^n - W_\varepsilon^{n-1}}{\Delta t_n}\|_{(H^1(\Omega))'}^2 + \|\tfrac{Z_\varepsilon^n - Z_\varepsilon^{n-1}}{\Delta t_n}\|_{(H^1(\Omega))'}^2 \right]$$
$$+ \sum_{n=1}^N \Delta t_n \left[\|\widetilde{\mathcal{G}}[\tfrac{W_\varepsilon^n - W_\varepsilon^{n-1}}{\Delta t_n}]\|_1^2 + \|\mathcal{G}[\tfrac{Z_\varepsilon^n - Z_\varepsilon^{n-1}}{\Delta t_n}]\|_1^2 \right] \leq C\,. \tag{6.3.29}$$

Proof: We consider the case when $r < \frac{\mu^2}{2}$ and we comment later on the simple case $r \geq \frac{\mu^2}{2}$. First of all, we note from (6.3.23) and (6.3.25) that

$$(W_\varepsilon^n, 1) = (W_\varepsilon^0, 1) \leq C \qquad n = 1, \cdots, N. \tag{6.3.30}$$

Using (2.4.2), (2.1.8), the stated choice $r = \frac{\rho\,\lambda}{3\,C_p}$, the assumption (iii) and (6.3.30) we obtain for $n = 2, \cdots, N$, that

$$\tfrac{r}{2}\,(1-\theta)\Delta t_n\,|W_\varepsilon^{n-1}|_h^2 \leq \tfrac{\rho\,\lambda}{2}\,(1-\theta)\Delta t_{n-1}|W_\varepsilon^{n-1}|_1^2 + C\,\Delta t_{n-1}\,. \tag{6.3.31}$$

Since $r < \frac{\mu^2}{2}$ and $\Phi_\varepsilon(s) \geq 0$, we have from (6.3.21) and (6.3.30) for $n = 1, \cdots, N$ that

$$\left(\tfrac{1}{2} - (\tfrac{\mu^2}{2\,r} - 1)\,\Delta t_n\right) \left[2\,(\Phi_\varepsilon(W_\varepsilon^n),1)^h + |Z_\varepsilon^n|_h^2 \right] + \rho\,\lambda\,(1-\tfrac{\theta}{2})\,\Delta t_n\,|W_\varepsilon^n|_1^2 + \Delta t_n\,|Z_\varepsilon^n|_1^2$$
$$\leq \left(\tfrac{1}{2} + (\tfrac{\mu^2}{2\,r} - 1)\,\Delta t_n\right) \left[2\,(\Phi_\varepsilon(W_\varepsilon^{n-1}),1)^h + |Z_\varepsilon^{n-1}|_h^2 \right]$$
$$+ \tfrac{r}{2}\,(1-\theta)\,\Delta t_n\,|W_\varepsilon^{n-1}|_h^2 + C\,\Delta t_n\,. \tag{6.3.32}$$

It follows from (6.3.32) and the assumption (ii) , for $n = 1, \cdots, N$, that

$$
\begin{aligned}
&2\,(\Phi_\varepsilon(W_\varepsilon^n), 1)^h + |Z_\varepsilon^n|_h^2 + \tfrac{\rho\,\lambda\,(1-\frac{\theta}{2})\,\Delta t_n}{\frac{1}{2}-(\frac{\mu^2}{2\,r}-1)\,\Delta t_n}\,|W_\varepsilon^n|_1^2 \\
&\qquad \le \left(1 + \tfrac{2}{\delta}\,(\tfrac{\mu^2}{2\,r} - 1)\,\Delta t_n\right) \left[\,2\,(\Phi_\varepsilon(W_\varepsilon^{n-1}), 1)^h + |Z_\varepsilon^{n-1}|_h^2\,\right] \\
&\qquad\qquad\qquad\qquad + \tfrac{\frac{r}{2}\,(1-\theta)\,\Delta t_n}{\frac{1}{2}-(\frac{\mu^2}{2\,r}-1)\,\Delta t_n}\,|W_\varepsilon^{n-1}|_h^2 + \tfrac{C}{\delta}\,\Delta t_n \\
&\qquad \le e^{\frac{2}{\delta}\,(\frac{\mu^2}{2\,r}-1)\,\Delta t_n} \left[\,2\,(\Phi_\varepsilon(W_\varepsilon^{n-1}), 1)^h + |Z_\varepsilon^{n-1}|_h^2 + \tfrac{\frac{r}{2}\,(1-\theta)\,\Delta t_n}{\frac{1}{2}-(\frac{\mu^2}{2\,r}-1)\,\Delta t_n}\,|W_\varepsilon^{n-1}|_h^2\right] + \tfrac{C}{\delta}\,\Delta t_n\,.
\end{aligned}
\tag{6.3.33}
$$

Observing (6.3.31) and the assumptions (ii) and (iii) yields for $n = 2, \cdots, N$ that

$$
\tfrac{\frac{r}{2}\,(1-\theta)\,\Delta t_n}{\frac{1}{2}-(\frac{\mu^2}{2\,r}-1)\,\Delta t_n}\,|W_\varepsilon^{n-1}|_h^2 \le \tfrac{\rho\,\lambda\,(1-\frac{\theta}{2})\,\Delta t_{n-1}}{\frac{1}{2}-(\frac{\mu^2}{2\,r}-1)\,\Delta t_{n-1}}\,|W_\varepsilon^{n-1}|_1^2 + \tfrac{C}{\delta}\,\Delta t_{n-1}\,. \tag{6.3.34}
$$

Substituting (6.3.34) into (6.3.33) and noting (2.4.2) and (6.3.25) leads to

$$
\begin{aligned}
\max_{n=1,\cdots,N}\left[\,(\Phi_\varepsilon(W_\varepsilon^n), 1)^h + \|Z_\varepsilon^n\|_0^2\,\right] &\le C\,e^{\frac{2}{\delta}\,(\frac{\mu^2}{2\,r}-1)\,T}\left[\,T + (\Phi_\varepsilon(W_\varepsilon^0), 1)^h + \|Z_\varepsilon^0\|_0^2 + \|W_\varepsilon^0\|_0^2\,\right] \\
&\le C\,.
\end{aligned}
\tag{6.3.35}
$$

Therefore, the first and the third bounds in (6.3.27) follow from (6.3.35). The second bound in (6.3.27) follows immediately from the first bound in (6.3.27) and (6.2.3b). The last two bounds in (6.3.27) can be obtained easily by summing (6.3.21) over n on noting (6.3.25), (6.3.30) and the third bound in (6.3.27). When $r \le \frac{\mu^2}{2}$, the result (6.3.27) follows directly from (6.3.21), (6.2.3b), (6.3.25) and (6.3.31).

Now, the result (6.3.28) can be easily established from the first bound in (6.3.27) and (6.2.7) on noting the equivalence (2.4.2).

To complete the proof it is still to show (6.3.29). From (3.1.1), (6.3.6a,b), (6.3.3), (3.1.3) and (2.4.2) we obtain for any $\eta \in H^1(\Omega)$ and for $n = 1, \cdots, N$ that

$$
\begin{aligned}
\langle \tfrac{W_\varepsilon^n - W_\varepsilon^{n-1}}{\Delta t_n}, \eta\rangle = \left(\tfrac{W_\varepsilon^n - W_\varepsilon^{n-1}}{\Delta t_n}, \eta\right) &= \left(\tfrac{W_\varepsilon^n - W_\varepsilon^{n-1}}{\Delta t_n}, P^h\eta\right)^h \\
&= \lambda\,\left(\Pi_\varepsilon(W_\varepsilon^n)\,\nabla Z_\varepsilon^n, \nabla P^h\eta\right) - \rho\,\left(\nabla W_\varepsilon^n, \nabla P^h\eta\right) \\
&\le C\,\left(|W_\varepsilon^n|_1 + |Z_\varepsilon^n|_1\right)\,|P^h\eta|_1 \\
&\le C\,\left(\|W_\varepsilon^n\|_1 + \|Z_\varepsilon^n\|_1\right)\,\|\eta\|_1
\end{aligned}
\tag{6.3.36a}
$$

and

$$
\begin{aligned}
\langle \tfrac{Z_\varepsilon^n - Z_\varepsilon^{n-1}}{\Delta t_n}, \eta \rangle = \left(\tfrac{Z_\varepsilon^n - Z_\varepsilon^{n-1}}{\Delta t_n}, \eta \right) &= \left(\tfrac{Z_\varepsilon^n - Z_\varepsilon^{n-1}}{\Delta t_n}, P^h \eta \right)^h \\
&= \mu \, \left(\theta \, W_\varepsilon^n + (1-\theta) \, W_\varepsilon^{n-1}, P^h \eta \right)^h - \left(Z_\varepsilon^n, P^h \eta \right)^h \\
&\qquad - \left(\nabla Z_\varepsilon^n, \nabla P^h \eta \right) - \lambda \, \left(\nabla W_\varepsilon^n, \nabla P^h \eta \right) \\
&\leq C \, \left(\|W_\varepsilon^n\|_1 + \|Z_\varepsilon^n\|_1 + \|W_\varepsilon^{n-1}\|_0 \right) \, \|\eta\|_1 \,. \qquad (6.3.36b)
\end{aligned}
$$

Hence, the desired result (6.3.29) follows from (6.3.36a,b), the assumption (iii), (6.3.27), (6.3.25), (6.3.9), (2.1.8) and (3.1.9). □

In the next subsection we exploit the uniform bounds derived in Theorem 6.3.6 on the approximations, independently of the parameters h, Δt and ε, to prove uniqueness of the approximations for sufficiently small time discretization parameter.

6.3.4 Uniqueness of the approximation

Theorem 6.3.7 Let the assumptions of Theorem 6.3.6 hold. Let $\{W_\varepsilon^n, Z_\varepsilon^n\}_{n=1}^N$ be a solution of the problem $(\mathrm{Q}_\varepsilon^{h,\Delta t})$ such that

$$\max_{n=1,\cdots,N} \|Z_\varepsilon^n\|_0 \leq C_b \,,$$

where C_b is a positive constant independent of the parameters h, Δt and ε. Then, for sufficiently small Δt, the solution $\{W_\varepsilon^n, Z_\varepsilon^n\}_{n=1}^N$ is unique.

Proof: Assume there are two discrete solutions $\{W_{\varepsilon,1}^n, Z_{\varepsilon,1}^n\}_{n=1}^N$ and $\{W_{\varepsilon,2}^n, Z_{\varepsilon,2}^n\}_{n=1}^N$ to the problem $(\mathrm{Q}_\varepsilon^{h,\Delta t})$ such that

$$\max_{n=1,\cdots,N} \left\{ \, \|Z_{\varepsilon,1}^n\|_0 \,, \, \|Z_{\varepsilon,2}^n\|_0 \, \right\} \leq C_b \,. \qquad (6.3.37)$$

We perform the proof by induction. On noting that we have uniqueness at time $t = 0$, we assume uniqueness of the approximations at the $(n-1)$-time step of $(\mathrm{Q}_\varepsilon^{h,\Delta t})$. Now, setting $\mathcal{W}_\varepsilon^n := W_{\varepsilon,1}^n - W_{\varepsilon,2}^n$ and $\mathcal{Z}_\varepsilon^n := Z_{\varepsilon,1}^n - Z_{\varepsilon,2}^n$, and subtracting the fully discrete approximations yields for all $\chi \in S^h$ that

$$\tfrac{1}{\Delta t_n}\,(\mathcal{W}^n_\varepsilon,\chi)^h + \rho\,(\nabla \mathcal{W}^n_\varepsilon,\nabla\chi) = \lambda\,\big(\Pi_\varepsilon(W^n_{\varepsilon,1})\,\nabla Z^n_{\varepsilon,1} - \Pi_\varepsilon(W^n_{\varepsilon,2})\,\nabla Z^n_{\varepsilon,2}\,,\nabla\chi\big)\,, \tag{6.3.38a}$$

$$\tfrac{1}{\Delta t_n}\,(\mathcal{Z}^n_\varepsilon,\chi)^h + (\mathcal{Z}^n_\varepsilon,\chi)^h + (\nabla\mathcal{Z}^n_\varepsilon,\nabla\chi) + \lambda\,(\nabla\mathcal{W}^n_\varepsilon,\nabla\chi) = \mu\,\theta\,(\mathcal{W}^n_\varepsilon,\chi)^h\,. \tag{6.3.38b}$$

Choosing $\chi \equiv \mathcal{W}^n_\varepsilon$ in (6.3.38a) and $\chi \equiv \frac{1}{\lambda}\,\mathcal{Z}^n_\varepsilon$ in (6.3.38b) and adding the resulting equations yields, on using the Hölder's inequality, (6.3.3), (6.3.4), (2.4.14) and (6.3.37), that

$$\begin{aligned}
&\tfrac{1}{\Delta t_n}\,|\mathcal{W}^n_\varepsilon|_h^2 + \tfrac{1}{\lambda\,\Delta t_n}\,|\mathcal{Z}^n_\varepsilon|_h^2 + \tfrac{1}{\lambda}\,|\mathcal{Z}^n_\varepsilon|_h^2 + \rho\,|\mathcal{W}^n_\varepsilon|_1^2 + \tfrac{1}{\lambda}\,|\mathcal{Z}^n_\varepsilon|_1^2 \\
&\quad = \lambda\,\big(\Pi_\varepsilon(W^n_{\varepsilon,1})\,\nabla Z^n_{\varepsilon,1} - \Pi_\varepsilon(W^n_{\varepsilon,2})\,\nabla Z^n_{\varepsilon,2}\,,\nabla\mathcal{W}^n_\varepsilon\big) - (\nabla\mathcal{W}^n_\varepsilon,\nabla\mathcal{Z}^n_\varepsilon) + \tfrac{\mu\,\theta}{\lambda}\,(\mathcal{W}^n_\varepsilon,\mathcal{Z}^n_\varepsilon)^h \\
&\quad = \lambda\,\big(\,\big[\Pi_\varepsilon(W^n_{\varepsilon,1}) - \tfrac{1}{\lambda}\big]\,\nabla\mathcal{Z}^n_\varepsilon,\nabla\mathcal{W}^n_\varepsilon\big) + \lambda\,\big(\,\big[\Pi_\varepsilon(W^n_{\varepsilon,1}) - \Pi_\varepsilon(W^n_{\varepsilon,2})\,\big]\,\nabla Z^n_{\varepsilon,2}\,,\nabla\mathcal{W}^n_\varepsilon\big) \\
&\qquad\qquad + \tfrac{\mu\,\theta}{\lambda}\,(\mathcal{W}^n_\varepsilon,\mathcal{Z}^n_\varepsilon)^h \\
&\quad \le |\mathcal{W}^n_\varepsilon|_1\,|\mathcal{Z}^n_\varepsilon|_1 + \tfrac{4\,C_1\,C_b}{\varepsilon\,h}\,\|\mathcal{W}^n_\varepsilon\|_{0,\infty}\,|\mathcal{W}^n_\varepsilon|_1 + \tfrac{\mu\,\theta}{\lambda}\,|\mathcal{W}^n_\varepsilon|_h\,|\mathcal{Z}^n_\varepsilon|_h \\
&\quad := I_1 + I_2 + I_3\,,
\end{aligned} \tag{6.3.39}$$

where

$$\begin{aligned}
I_1 &:= |\mathcal{W}^n_\varepsilon|_1\,|\mathcal{Z}^n_\varepsilon|_1\,,\\
I_2 &:= \tfrac{4\,C_1\,C_b}{\varepsilon\,h}\,\|\mathcal{W}^n_\varepsilon\|_{0,\infty}\,|\mathcal{W}^n_\varepsilon|_1\,,\\
I_3 &:= \tfrac{\mu\,\theta}{\lambda}\,|\mathcal{W}^n_\varepsilon|_h\,|\mathcal{Z}^n_\varepsilon|_h\,,
\end{aligned}$$

and C_1 is the positive constant, independent of the parameters h, Δt and ε, that is generated from applying (2.4.14).

It follows from the Young's inequality, (2.4.14) and (2.4.15) that

$$I_1 \le \rho\,|\mathcal{W}^n_\varepsilon|_1^2 + \tfrac{1}{4\,\rho}\,|\mathcal{Z}^n_\varepsilon|_1^2 \le \rho\,|\mathcal{W}^n_\varepsilon|_1^2 + \tfrac{C_1^2}{4\,\rho\,h^2}\,\|\mathcal{Z}^n_\varepsilon\|_0^2\,, \tag{6.3.40a}$$

$$I_2 \le \tfrac{4\,C_1^2\,C_2\,C_b}{\varepsilon\,h^{\frac{5}{2}}}\,\|\mathcal{W}^n_\varepsilon\|_0^2\,, \tag{6.3.40b}$$

$$I_3 \le \tfrac{\mu^2\,\theta^2}{4\,\lambda}\,|\mathcal{W}^n_\varepsilon|_h^2 + \tfrac{1}{\lambda}\,|\mathcal{Z}^n_\varepsilon|_h^2\,, \tag{6.3.40c}$$

where C_2 is the positive constant, independent of h, Δt and ε, generated from applying (2.4.15). Combining (6.3.39) and (6.3.40a)-(6.3.40c) yields on noting the equivalence (2.4.2) that

$$\left(\tfrac{1}{\Delta t_n} - \left[\tfrac{4\,C_1^2\,C_2\,C_b}{\varepsilon\,h^{\frac{5}{2}}} + \tfrac{\mu^2\,\theta^2}{4\,\lambda}\right]\right) |\mathcal{W}_\varepsilon^n|_h^2 + \left(\tfrac{1}{\lambda\,\Delta t_n} - \tfrac{C_1^2}{4\,\rho\,h^2}\right) |\mathcal{Z}_\varepsilon^n|_h^2 \leq 0\,. \tag{6.3.41}$$

Alternatively to (6.3.40a)-(6.3.40c), we have from the Young's inequality, (2.4.14) and (2.4.15) that

$$I_1 \leq \tfrac{\lambda}{4}\,|\mathcal{W}_\varepsilon^n|_1^2 + \tfrac{1}{\lambda}\,|\mathcal{Z}_\varepsilon^n|_1^2 \leq \tfrac{\lambda\,C_1^2}{4\,h^2}\,\|\mathcal{W}_\varepsilon^n\|_0^2 + \tfrac{1}{\lambda}\,|\mathcal{Z}_\varepsilon^n|_1^2\,, \tag{6.3.42a}$$

$$I_2 \leq \tfrac{4\,C_1\,C_2\,C_b}{\varepsilon\,h^{\frac{3}{2}}}\,\|\mathcal{W}_\varepsilon^n\|_0\,|\mathcal{W}_\varepsilon^n|_1 \leq \tfrac{(2\,C_1\,C_2\,C_b)^2}{\rho\,\varepsilon^2\,h^3}\,\|\mathcal{W}_\varepsilon^n\|_0^2 + \rho\,|\mathcal{W}_\varepsilon^n|_1^2\,, \tag{6.3.42b}$$

$$I_3 \leq \tfrac{\mu^2\,\theta^2\,\Delta t_n}{4\,\lambda}\,|\mathcal{W}_\varepsilon^n|_h^2 + \tfrac{1}{\lambda\,\Delta t_n}\,|\mathcal{Z}_\varepsilon^n|_h^2\,. \tag{6.3.42c}$$

Putting (6.3.42a)-(6.3.42c) in (6.3.39) and noting (2.4.2) gives that

$$\left(\tfrac{1}{\Delta t_n} - \left[\tfrac{\lambda\,C_1^2}{4\,h^2} + \tfrac{(2\,C_1\,C_2\,C_b)^2}{\rho\,\varepsilon^2\,h^3} + \tfrac{\mu^2\,\theta^2\,\Delta t_n}{4\,\lambda}\right]\right) |\mathcal{W}_\varepsilon^n|_h^2 + \tfrac{1}{\lambda}\,|\mathcal{Z}_\varepsilon^n|_h^2 \leq 0\,. \tag{6.3.43}$$

Suppose that $\lambda < 4\,\rho$, we obtain from the Young's inequality and (2.4.15) that

$$I_1 \leq \tfrac{\lambda}{4}\,|\mathcal{W}_\varepsilon^n|_1^2 + \tfrac{1}{\lambda}\,|\mathcal{Z}_\varepsilon^n|_1^2\,, \tag{6.3.44a}$$

$$I_2 \leq \tfrac{4\,C_1\,C_2\,C_b}{\varepsilon\,h^{\frac{3}{2}}}\,\|\mathcal{W}_\varepsilon^n\|_0\,|\mathcal{W}_\varepsilon^n|_1 \leq \tfrac{(4\,C_1\,C_2\,C_b)^2}{(4\,\rho-\lambda)\,\varepsilon^2\,h^3}\,\|\mathcal{W}_\varepsilon^n\|_0^2 + \tfrac{4\,\rho-\lambda}{4}\,|\mathcal{W}_\varepsilon^n|_1^2\,, \tag{6.3.44b}$$

$$I_3 \leq \tfrac{\mu^2\,\theta^2\,\Delta t_n}{4\,\lambda}\,|\mathcal{W}_\varepsilon^n|_h^2 + \tfrac{1}{\lambda\,\Delta t_n}\,|\mathcal{Z}_\varepsilon^n|_h^2\,. \tag{6.3.44c}$$

From (6.3.39) and (6.3.42a)-(6.3.42c) we have, when $\lambda < 4\,\rho$, that

$$\left(\tfrac{1}{\Delta t_n} - \left[\tfrac{(4\,C_1\,C_2\,C_b)^2}{(4\,\rho-\lambda)\,\varepsilon^2\,h^3} + \tfrac{\mu^2\,\theta^2\,\Delta t_n}{4\,\lambda}\right]\right) |\mathcal{W}_\varepsilon^n|_h^2 + \tfrac{1}{\lambda}\,|\mathcal{Z}_\varepsilon^n|_h^2 \leq 0\,. \tag{6.3.45}$$

Now, we set

$$\tau_1 := \min\left\{ \left(\tfrac{4\,C_1^2\,C_2\,C_b}{\varepsilon\,h^{\frac{5}{2}}} + \tfrac{\mu^2\,\theta^2}{4\,\lambda}\right)^{-1},\ \tfrac{4\,\rho\,h^2}{\lambda\,C_1^2} \right\},$$

$$\tau_2 := \begin{cases} \left(\tfrac{\lambda\,C_1^2}{4\,h^2} + \tfrac{(2\,C_1\,C_2\,C_b)^2}{\rho\,\varepsilon^2\,h^3}\right)^{-1} & \text{if } \mu = 0 \text{ or } \theta = 0\,, \\ \tfrac{2\,\lambda}{\mu^2\,\theta^2}\left(\left[\left(\tfrac{\lambda\,C_1^2}{4\,h^2} + \tfrac{(2\,C_1\,C_2\,C_b)^2}{\rho\,\varepsilon^2\,h^3}\right)^2 + \tfrac{\mu^2\,\theta^2}{\lambda}\right]^{\frac{1}{2}} - \left[\tfrac{\lambda\,C_1^2}{4\,h^2} + \tfrac{(2\,C_1\,C_2\,C_b)^2}{\rho\,\varepsilon^2\,h^3}\right]\right) & \\ & \text{if } \mu \neq 0 \text{ and } \theta \neq 0 \end{cases}$$

and

$$\tau_3 := \begin{cases} 0 & \text{if } \lambda \geq 4\,\rho\,, \\ \frac{(4\,\rho-\lambda)\,\varepsilon^2\,h^3}{(4\,C_1\,C_2\,C_b)^2} & \text{if } \lambda < 4\,\rho \text{ and } (\mu = 0 \text{ or } \theta = 0)\,, \\ \frac{2\,\lambda}{\mu^2\,\theta^2}\left(\left[\left(\frac{(4\,C_1\,C_2\,C_b)^2}{(4\,\rho-\lambda)\,\varepsilon^2\,h^3}\right)^2 + \frac{\mu^2\,\theta^2}{\lambda}\right]^{\frac{1}{2}} - \frac{(4\,C_1\,C_2\,C_b)^2}{(4\,\rho-\lambda)\,\varepsilon^2\,h^3}\right) & \\ & \text{if } \lambda < 4\,\rho \text{ and } (\mu \neq 0 \text{ and } \theta \neq 0)\,. \end{cases}$$

On noting (6.3.41), (6.3.43) and (6.3.45), we obtain for any $\Delta t \in (0, \max\{\tau_1\,,\,\tau_2\,,\,\tau_3\})$ that

$$|\mathcal{W}^n_\varepsilon|^2_h + |\mathcal{Z}^n_\varepsilon|^2_h \leq 0 \qquad n \geq 1\,.$$

We thus conclude $W^n_{\varepsilon,1} \equiv W^n_{\varepsilon,2}$ and $Z^n_{\varepsilon,1} \equiv Z^n_{\varepsilon,2}$ for all $n \geq 1$ as required. □

6.4 Existence and uniqueness of a weak solution

By extending the notation (3.3.1a)-(3.3.3a) to W_ε and Z_ε and noting (6.3.6a,b), we can rewrite the problem $(\mathrm{Q}^{h,\Delta t}_\varepsilon)$ as:

Find $\{W_\varepsilon,\,Z_\varepsilon\} \in C([0,T];S^h) \times C([0,T];S^h)$ such that for all $\chi \in L^2(0,T;S^h)$

$$\int_0^T \left[\left(\tfrac{\partial W_\varepsilon}{\partial t}, \chi\right)^h + \rho\,(\,\nabla W^+_\varepsilon, \nabla\chi)\right]\,\mathrm{d}t \;=\lambda \int_0^T (\,\Pi_\varepsilon(W^+_\varepsilon)\,\nabla Z^+_\varepsilon, \nabla\chi)\,\mathrm{d}t\,, \tag{6.4.1a}$$

$$\int_0^T \left[\left(\tfrac{\partial Z_\varepsilon}{\partial t}, \chi\right)^h + \left(Z^+_\varepsilon, \chi\right)^h + (\,\nabla Z^+_\varepsilon, \nabla\chi) + \lambda\,(\,\nabla W^+_\varepsilon, \nabla\chi)\right]\,\mathrm{d}t$$
$$= \mu \int_0^T \left(\theta\,W^+_\varepsilon + (1-\theta)\,W^-_\varepsilon, \chi\right)^h\,\mathrm{d}t\,. \tag{6.4.1b}$$

Theorem 6.4.1 Let all the assumptions in Theorem 6.3.6 hold and let $w^0,\ z^0 \in H^1(\Omega)$ with $|w^0(\cdot)| \leq 1$ *a.e.* in Ω. Let $W^0_\varepsilon \equiv P^h w^0$, $Z^0_\varepsilon \equiv P^h z^0$; or $W^0_\varepsilon \equiv \pi^h w^0$, $Z^0_\varepsilon \equiv \pi^h z^0$. In addition to the assumptions (i)-(iii) in Theorem 6.3.6 , let

(iv) $\Delta t\,,\ \varepsilon\ \to 0$ as $h\ \to 0\,.$

Then there exists a subsequence of $\{W_\varepsilon, Z_\varepsilon\}_{h>0}$, where $\{W_\varepsilon, Z_\varepsilon\}$ solves $(\mathrm{Q}^{h,\Delta t}_\varepsilon)$, and functions

$$w\,,\,z \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;(H^1(\Omega))') \tag{6.4.2a}$$

satisfy

$$w(\cdot,0) = w^0(\cdot) \quad \text{and} \quad z(\cdot,0) = z^0(\cdot) \quad \text{in} \quad L^2(\Omega). \tag{6.4.2b}$$

In addition,

$$|w(x,t)| \leq 1 \qquad a.e. \text{ in } \Omega_T\,. \tag{6.4.2c}$$

Furthermore, it holds as $h \to 0$ that

$$W_\varepsilon,\ W_\varepsilon^\pm \rightharpoonup w \quad \text{and} \quad Z_\varepsilon,\ Z_\varepsilon^\pm \rightharpoonup z \quad \text{in} \quad L^2(0,T;H^1(\Omega))\,, \tag{6.4.3a}$$

$$W_\varepsilon,\ W_\varepsilon^\pm \rightharpoonup^* w \quad \text{and} \quad Z_\varepsilon,\ Z_\varepsilon^\pm \rightharpoonup^* z \quad \text{in} \quad L^\infty(0,T;L^2(\Omega))\,, \tag{6.4.3b}$$

$$\frac{\partial W_\varepsilon}{\partial t} \rightharpoonup \frac{\partial w}{\partial t} \quad \text{and} \quad \frac{\partial Z_\varepsilon}{\partial t} \rightharpoonup \frac{\partial z}{\partial t} \quad \text{in} \quad L^2(0,T;(H^1(\Omega))')\,, \tag{6.4.3c}$$

$$W_\varepsilon,\ W_\varepsilon^\pm \to w \quad \text{and} \quad Z_\varepsilon,\ Z_\varepsilon^\pm \to z \quad \text{in} \quad L^2(0,T;L^\infty(\Omega)) \tag{6.4.3d}$$

and

$$\Pi_\varepsilon(W_\varepsilon^+) \to \mathcal{V}(w) \quad \text{in} \quad L^2(0,T;L^\infty(\Omega))\,. \tag{6.4.3e}$$

Proof: The proof is obtained using a sequential compactness argument, see Theorem 3.3.1 where a similar argument is employed. First of all, we note from (3.1.3), (2.4.16) and the stated assumptions on the initial data that

$$\|W_\varepsilon^0\|_1 + \|Z_\varepsilon^0\|_1 \leq C\,, \tag{6.4.4}$$

and

$$W_\varepsilon^0 \to w_0 \quad \text{and} \quad Z_\varepsilon^0 \to z_0 \qquad \text{in } L^2(\Omega). \tag{6.4.5}$$

It follows from (6.3.27)-(6.3.29), (6.4.4), (6.2.5) and (6.3.3) that

$$\begin{aligned}
&\|W_\varepsilon^{(\pm)}\|_{L^2(0,T;H^1(\Omega))} + \|W_\varepsilon^{(\pm)}\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{-\frac{1}{2}}\|\pi^h[W_\varepsilon^{(\pm)} - 1]_+\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad + \varepsilon^{-\frac{1}{2}}\|\pi^h[-1 - W_\varepsilon^{(\pm)}]_+\|_{L^\infty(0,T;L^2(\Omega))} + \|\tfrac{\partial W_\varepsilon}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} \\
&\quad + \|\widetilde{\mathcal{G}}\tfrac{\partial W_\varepsilon}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\mathcal{V}_\varepsilon(W_\varepsilon^+)\|_{L^\infty(\Omega_T)} + \|\Pi_\varepsilon(W_\varepsilon^+)\|_{L^\infty(\Omega_T)} \leq C\,,
\end{aligned} \tag{6.4.6a}$$

and

$$\|Z_\varepsilon^{(\pm)}\|_{L^2(0,T;H^1(\Omega))} + \|Z_\varepsilon^{(\pm)}\|_{L^\infty(0,T;L^2(\Omega))} \\ + \|\tfrac{\partial Z_\varepsilon}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} + \|\mathcal{G}\tfrac{\partial Z_\varepsilon}{\partial t}\|_{L^2(0,T;H^1(\Omega))} \leq C\,. \tag{6.4.6b}$$

Furthermore, we have from the fifth bound in (6.4.6a) and the third bound in (6.4.6b) that

$$\|W_\varepsilon^\pm - W_\varepsilon\|^2_{L^2(0,T;(H^1(\Omega))')} + \|Z_\varepsilon^\pm - Z_\varepsilon\|^2_{L^2(0,T;(H^1(\Omega))')} \\ \leq (\Delta t)^2\,\|\tfrac{\partial W_\varepsilon}{\partial t}\|^2_{L^2(0,T;(H^1(\Omega))')} + (\Delta t)^2\,\|\tfrac{\partial Z_\varepsilon}{\partial t}\|^2_{L^2(0,T;(H^1(\Omega))')} \leq C\,(\Delta t)^2\,. \tag{6.4.7}$$

From (6.4.6a,b), (6.4.7), (2.1.4) and the compact embedding $H^1(\Omega) \overset{c}{\hookrightarrow} L^\infty(\Omega)$, one can obtain using sequential compactness arguments the existence of a subsequence of $\{W_\varepsilon, Z_\varepsilon\}_h$, still denoted $\{W_\varepsilon, Z_\varepsilon\}_h$, and functions $\{w, z\}$ such that the results (6.4.2a) and (6.4.3a)-(6.4.3d) hold, see the proof of Theorem 3.3.1 for instance. As

$$W_\varepsilon\,,\; Z_\varepsilon\,,\; w\,,\; z \in \left\{\eta :\; \eta \in L^2(0,T;H^1(\Omega)),\; \tfrac{\partial \eta}{\partial t} \in L^2(0,T;(H^1(\Omega))')\right\},$$

we have from Theorem 7.2 in Robinson [58] that

$$W_\varepsilon\,,\; Z_\varepsilon\,,\; w\,,\; z \in C([0,T];L^2(\Omega))\,. \tag{6.4.8}$$

Thus, (6.4.2b) follows from (6.4.3d), (6.4.5) and (6.4.8). The bound (6.4.2c) follows immediately from the third and the fourth bounds in (6.4.6a) and the strong convergence (6.4.3d).

It is still to show the convergence result (6.4.3e). To do so, we first note from (6.3.5), (2.4.15) and the first bound in (6.4.6a) that

$$\left\|\Pi_\varepsilon(W_\varepsilon^+) - \mathcal{V}_\varepsilon(W_\varepsilon^+)\right\|_{L^2(0,T;L^\infty(\Omega))} \leq C\,h^{\frac{1}{2}}\|W_\varepsilon^+\|_{L^2(0,T;H^1(\Omega))} \leq C\,h^{\frac{1}{2}} \to 0 \;\text{ as }\; h \to 0\,. \tag{6.4.9}$$

From the Lipschitz continuity of the function $\mathcal{V}_\varepsilon(s)$ on $\mathbb{R}$, (6.4.2c), (6.2.6), (6.4.3d) and the assumption (iv) we have that

$$\left\|\mathcal{V}_\varepsilon(W_\varepsilon^+) - \mathcal{V}_\varepsilon(w)\right\|_{L^2(0,T;L^\infty(\Omega))} + \|\mathcal{V}_\varepsilon(w) - \mathcal{V}(w)\|_{L^2(0,T;L^\infty(\Omega))} \\ \leq C\,\left(\left\|W_\varepsilon^+ - w\right\|_{L^2(0,T;L^\infty(\Omega))} + \varepsilon\right) \to 0 \;\text{ as }\; h \to 0\,. \tag{6.4.10}$$

Thus, (6.4.3e) follows by combining (6.4.9) and (6.4.10). □

We now show the main theorem in this chapter which deals with the existence and the uniqueness of a global weak solution to the system (1.2.1a)-(1.2.1d).

Theorem 6.4.2 Let the assumptions of Theorem 6.4.1 hold. Then there exists a subsequence of $\{W_\varepsilon, Z_\varepsilon\}_{h>0}$, where $\{W_\varepsilon, Z_\varepsilon\}$ solves $(\mathrm{Q}_\varepsilon^{h,\Delta t})$, and functions $\{w, z\}$ satisfying (6.4.2a)-(6.4.2c) and $⨍\, w = ⨍\, w^0$ for *a.e.* $t \in (0,T)$. In addition, as $h \to 0$ the convergence results (6.4.3a)-(6.4.3e) hold. Furthermore, the functions $\{w, z\}$ represent a global weak solution of the problem (Q) in sense that for all $\eta \in L^2(0,T;H^1(\Omega))$

$$\int_0^T \left[\langle \tfrac{\partial w}{\partial t}, \eta\rangle + \rho\,(\nabla w, \nabla \eta)\right]\,\mathrm{d}t \;=\int_0^T (\,[1-w^2]\,\nabla z, \nabla\eta)\,\mathrm{d}t\,, \qquad (6.4.11a)$$

$$\int_0^T \left[\langle \tfrac{\partial z}{\partial t}, \eta\rangle + (z,\eta) + (\,\nabla z, \nabla \eta) + \lambda\,(\,\nabla w, \nabla \eta)\right]\,\mathrm{d}t \;= \mu\int_0^T (w,\eta)\,\mathrm{d}t\,. \qquad (6.4.11b)$$

Moreover, if $\lambda < 4\,\rho$ and the function z satisfies, additionally to (6.4.2a), that

$$\|z\|_{L^\infty(0,T;H^1(\Omega))} \le C\,, \qquad (6.4.12)$$

then the solution $\{w, z\}$ is unique.

Proof: We separate the proof into two parts. In the first part, Subsection 6.4.1, we briefly adapt the convergence arguments used in Theorem 3.3.3 to show that the functions $\{w, z\}$ defined by Theorem 6.4.1 represent a weak solution of problem (Q). In the second part, Subsection 6.4.2, we discuss the uniqueness of the weak solution.

6.4.1 Existence of a weak solution

For any $\eta \in L^2(0,T;H^1(\Omega))$, we set $\chi \equiv \pi^h\eta$ in (6.4.1a,b) and then we analyse the convergence of the resulting terms as $h \to 0$. On setting $Y_\varepsilon \equiv W_\varepsilon$ and Z_ε, respectively, with $\mathcal{G}_{W_\varepsilon} \equiv \widetilde{\mathcal{G}}$ and $\mathcal{G}_{Z_\varepsilon} \equiv \mathcal{G}$, we have from (2.4.19), the Hölder's inequality, the continuous embedding $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$, (6.3.11), (3.1.10), (2.4.16) and (6.4.6a,b) for all $\eta \in L^2(0,T;H^1(\Omega))$ and for all $\tilde{\eta} \in H^1(0,T;H^1(\Omega))$ that

$$
\begin{aligned}
\left| \int_0^T \left[\left(\tfrac{\partial Y_\varepsilon}{\partial t}, \pi^h \eta\right)^h - \left(\tfrac{\partial Y_\varepsilon}{\partial t}, \pi^h \eta\right) \right] \mathrm{d}t \right| \qquad & \\
\leq \left| \int_0^T \left[\left(\tfrac{\partial Y_\varepsilon}{\partial t}, \pi^h [\eta - \tilde{\eta}]\right)^h - \left(\tfrac{\partial Y_\varepsilon}{\partial t}, \pi^h [\eta - \tilde{\eta}]\right) \right] \mathrm{d}t \right| & \\
+ \left| \int_0^T \left[\left(Y_\varepsilon, \tfrac{\partial(\pi^h \tilde{\eta})}{\partial t}\right)^h - \left(Y_\varepsilon, \tfrac{\partial(\pi^h \tilde{\eta})}{\partial t}\right) \right] \mathrm{d}t \right| & \\
+ \left| (Y_\varepsilon(\cdot,T), \pi^h \tilde{\eta}(\cdot,T))^h - (Y_\varepsilon(\cdot,T), \pi^h \tilde{\eta}(\cdot,T)) \right| & \\
+ \left| (Y_\varepsilon(\cdot,0), \pi^h \tilde{\eta}(\cdot,0))^h - (Y_\varepsilon(\cdot,0), \pi^h \tilde{\eta}(\cdot,0)) \right| & \\
\leq C\, h \, \| \tfrac{\partial Y_\varepsilon}{\partial t} \|_{L^2(\Omega_T)} \, \| \pi^h [\eta - \tilde{\eta}] \|_{L^2(0,T;H^1(\Omega))} & \\
+ C\, h \, \| Y_\varepsilon \|_{L^\infty(0,T;L^2(\Omega))} \, \| \pi^h \tilde{\eta} \|_{H^1(0,T;H^1(\Omega))} & \\
\leq C \, \| \mathcal{G}_{Y_\varepsilon} \tfrac{\partial Y_\varepsilon}{\partial t} \|_{L^2(0,T;H^1(\Omega))} \, \| \eta - \tilde{\eta} \|_{L^2(0,T;H^1(\Omega))} + C\, h \, \| \tilde{\eta} \|_{H^1(0,T;H^1(\Omega))} & \\
\leq C \, \| \eta - \tilde{\eta} \|_{L^2(0,T;H^1(\Omega))} + C\, h \, \| \tilde{\eta} \|_{H^1(0,T;H^1(\Omega))}. & \qquad (6.4.13)
\end{aligned}
$$

We also have from the Hölder's inequality and (6.4.6a,b) for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$
\begin{aligned}
\left| \int_0^T \left(\tfrac{\partial Y_\varepsilon}{\partial t}, (\pi^h - I)\, \eta \right) \mathrm{d}t \right| &\leq \int_0^T \left| \left\langle \tfrac{\partial Y_\varepsilon}{\partial t}, (\pi^h - I)\, \eta \right\rangle \right| \mathrm{d}t \\
&\leq \left\| \tfrac{\partial Y_\varepsilon}{\partial t} \right\|_{L^2(0,T;(H^1(\Omega))')} \left\| (\pi^h - I)\, \eta \right\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C \, \left\| (\pi^h - I)\, \eta \right\|_{L^2(0,T;H^1(\Omega))}. \qquad (6.4.14)
\end{aligned}
$$

Combining (6.4.13), (6.4.14), the denseness of $H^1(0,T;H^1(\Omega))$ in $L^2(0,T;H^1(\Omega))$, (2.4.17) and (6.4.3c) yields for any $\eta \in L^2(0,T;H^1(\Omega))$ that

$$
\int_0^T \left(\tfrac{\partial Y_\varepsilon}{\partial t}, \pi^h \eta \right)^h \mathrm{d}t \quad \longrightarrow \int_0^T \left\langle \tfrac{\partial y}{\partial t}, \eta \right\rangle \mathrm{d}t \quad \text{as } h \to 0, \qquad (6.4.15)
$$

where $y \equiv w$ and z respectively.

Similarly to the derivation of (3.3.38) and (3.3.46), we can show on noting the bounds (6.4.6a,b) and the convergence result (6.4.3a), for any $\eta \in L^2(0,T;H^1(\Omega))$ that

$$
\int_0^T (\nabla Y_\varepsilon^+, \nabla \pi^h \eta) \, \mathrm{d}t \ \longrightarrow \int_0^T (\nabla y, \nabla \eta) \, \mathrm{d}t \quad \text{as } h \to 0 \qquad (6.4.16)
$$

and

$$\int_0^T (Y_\varepsilon^\pm, \pi^h \eta)^h \,\mathrm{d}t \;\longrightarrow\; \int_0^T (y, \eta)\,\mathrm{d}t \quad \text{as } h \to 0\,, \tag{6.4.17}$$

where $Y_\varepsilon^\pm \equiv W_\varepsilon^\pm$ and $Z_\varepsilon^\pm$, $y \equiv w$ and z respectively.

It follows from the Hölder's inequality and (6.4.6a,b) for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned}
&\left| \int_0^T \big(\Pi_\varepsilon(W_\varepsilon^+)\,\nabla Z_\varepsilon^+, \nabla(\pi^h - I)\,\eta\big)\ \mathrm{d}t \right| \\
&\qquad \le \left\|\Pi_\varepsilon(W_\varepsilon^+)\right\|_{L^\infty(\Omega_T)} \left\|Z_\varepsilon^+\right\|_{L^2(0,T;H^1(\Omega))} \left\|(\pi^h - I)\,\eta\right\|_{L^2(0,T;H^1(\Omega))} \\
&\qquad \le C\ \left\|(\pi^h - I)\,\eta\right\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned} \tag{6.4.18}$$

We also obtain from the Hölder's inequality, (6.4.6a,b), (6.4.2c) and (6.1.2) for all $\eta \in L^2(0,T;H^1(\Omega))$ and for all $\tilde{\eta} \in L^\infty(0,T;H^1(\Omega))$ that

$$\begin{aligned}
&\left| \int_0^T \big(\left[\Pi_\varepsilon(W_\varepsilon^+) - \mathcal{V}(w)\right] \nabla Z_\varepsilon^+, \nabla\eta\big)\ \mathrm{d}t \right| \\
&\quad \le \left| \int_0^T \big(\left[\Pi_\varepsilon(W_\varepsilon^+) - \mathcal{V}(w)\right] \nabla Z_\varepsilon^+, \nabla\left[\eta - \tilde{\eta}\right]\big)\ \mathrm{d}t \right| \\
&\qquad\qquad + \left| \int_0^T \big(\left[\Pi_\varepsilon(W_\varepsilon^+) - \mathcal{V}(w)\right] \nabla Z_\varepsilon^+, \nabla\tilde{\eta}\big)\ \mathrm{d}t \right| \\
&\quad \le \|\Pi_\varepsilon(W_\varepsilon^+) - \mathcal{V}(w)\|_{L^\infty(\Omega_T)}\, \|Z_\varepsilon^+\|_{L^2(0,T;H^1(\Omega))}\, \|\eta - \tilde{\eta}\|_{L^2(0,T;H^1(\Omega))} \\
&\qquad\qquad + \|\Pi_\varepsilon(W_\varepsilon^+) - \mathcal{V}(w)\|_{L^2(0,T;L^\infty(\Omega))}\, \|Z_\varepsilon^+\|_{L^2(0,T;H^1(\Omega))}\, \|\nabla\tilde{\eta}\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad \le C\,\|\eta - \tilde{\eta}\|_{L^2(0,T;H^1(\Omega))} + C\,\|\Pi_\varepsilon(W_\varepsilon^+) - \mathcal{V}(w)\|_{L^2(0,T;L^\infty(\Omega))}\, \|\tilde{\eta}\|_{L^\infty(0,T;H^1(\Omega))}.
\end{aligned} \tag{6.4.19}$$

Noting (6.4.18), (6.4.19), (2.4.17), the denseness of $L^\infty(0,T;H^1(\Omega))$ in $L^2(0,T;H^1(\Omega))$ and (6.4.3e) yields for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\int_0^T \big(\Pi_\varepsilon(W_\varepsilon^+)\,\nabla Z_\varepsilon^+, \nabla\pi^h\eta\big)\ \mathrm{d}t \;\to\; \int_0^T (\mathcal{V}(w)\,\nabla z, \nabla\eta)\ \mathrm{d}t \quad \text{as } h \to 0\,. \tag{6.4.20}$$

Now, we deduce from (6.4.1a)-(6.4.1b), (6.4.15)→(6.4.17), (6.4.20) and (6.1.2) that the functions $\{w, z\}$ satisfy (6.4.11a)-(6.4.11b), as well as the results of Theorem 6.4.1. Finally, we note from the weak formulation (6.4.11a) that $⨍\, w(\cdot,t) = ⨍\, w^0(\cdot)$ for *a.e.* $t \in (0,T)$. This completes the existence proof.

6.4.2 Uniqueness of the weak solution

We now show, under the assumption (6.4.12), the uniqueness of the weak solution if the cross diffusion coefficient λ is not too large. Essentially, the proof is identical to Galiano *et al.* [34] and is included for completeness.

Assume that there are two weak solutions $\{w_1, z_1\}$ and $\{w_2, z_2\}$ to the system (1.2.1a)-(1.2.1d) that satisfy (6.4.2a)-(6.4.2c) and (6.4.11a)-(6.4.11b). Further, assume that the functions z_1 and z_2 satisfy (6.4.12). As both solutions $\{w_1, z_1\}$ and $\{w_2, z_2\}$ satisfy (6.4.2b), we have that

$$w_1(\cdot, 0) = w_2(\cdot, 0) = w^0(\cdot) \quad \text{and} \quad z_1(\cdot, 0) = z_2(\cdot, 0) = z^0(\cdot) \quad \text{in} \quad L^2(\Omega). \tag{6.4.21}$$

Setting $\boldsymbol{w} := w_1 - w_2$, $\boldsymbol{z} := z_1 - z_2$ and testing (6.4.11a) with $\eta \equiv \boldsymbol{w} \in L^2(0, T; H^1(\Omega))$ and (6.4.11b) with $\eta \equiv \frac{1}{\lambda}\,\boldsymbol{z} \in L^2(0, T; H^1(\Omega))$ leads to after subtracting the weak forms

$$\begin{aligned}
\tfrac{1}{2}\,\|\boldsymbol{w}(T)\|_0^2 + \rho\,\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_T)} &= \tfrac{1}{2}\,\|\boldsymbol{w}(0)\|_0^2 + \int_0^T (\,\nabla \boldsymbol{z}, \nabla \boldsymbol{w})\ \mathrm{d}t \\
&\quad + \int_0^T (\,w_2^2\,\nabla z_2 - w_1^2\,\nabla z_1, \nabla \boldsymbol{w})\ \mathrm{d}t\,, \qquad (6.4.22\text{a}) \\
\tfrac{1}{2\,\lambda}\,\|\boldsymbol{z}(T)\|_0^2 + \tfrac{1}{\lambda}\,\|\boldsymbol{z}\|^2_{L^2(\Omega_T)} + \tfrac{1}{\lambda}\,\|\nabla \boldsymbol{z}\|^2_{L^2(\Omega_T)} &+ \int_0^T (\,\nabla \boldsymbol{w}, \nabla \boldsymbol{z})\ \mathrm{d}t \\
&= \tfrac{1}{2\,\lambda}\,\|\boldsymbol{z}(0)\|_0^2 + \tfrac{\mu}{\lambda}\int_0^T (\boldsymbol{w}, \boldsymbol{z})\ \mathrm{d}t\,. \qquad (6.4.22\text{b})
\end{aligned}$$

Adding (6.4.22a,b), noting (6.4.21) and employing the Hölder's inequality yields, on using (6.4.2c), that

$$\begin{aligned}
&\tfrac{1}{2}\,\left(\|\boldsymbol{w}(T)\|_0^2 + \tfrac{1}{\lambda}\,\|\boldsymbol{z}(T)\|_0^2\right) + \rho\,\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_T)} + \tfrac{1}{\lambda}\,\|\boldsymbol{z}\|^2_{L^2(\Omega_T)} + \tfrac{1}{\lambda}\,\|\nabla \boldsymbol{z}\|^2_{L^2(\Omega_T)} \\
&\quad = \tfrac{\mu}{\lambda}\int_0^T (\boldsymbol{w}, \boldsymbol{z})\ \mathrm{d}t\ - \int_0^T (\,w_1^2\,\nabla \boldsymbol{z}, \nabla \boldsymbol{w})\ \mathrm{d}t\ - \int_0^T (\,(w_1 + w_2)\,\boldsymbol{w}\,\nabla z_2, \nabla \boldsymbol{w})\ \mathrm{d}t \\
&\quad \leq \tfrac{\mu}{\lambda}\int_0^T \|\boldsymbol{w}\|_0\,\|\boldsymbol{z}\|_0\ \mathrm{d}t\ + \int_0^T |\boldsymbol{z}|_1\,|\boldsymbol{w}|_1\ \mathrm{d}t\ + 2\int_0^T \|\boldsymbol{w}\|_{0,\infty}\,|z_2|_1\,|\boldsymbol{w}|_1\ \mathrm{d}t\,.
\end{aligned} \tag{6.4.23}$$

We easily obtain from the Young's inequality that

$$\frac{\mu}{\lambda} \int_0^T \|\boldsymbol{w}\|_0 \, \|\boldsymbol{z}\|_0 \, \mathrm{d}t \;\leq \tfrac{\mu^2}{4\,\lambda} \, \|\boldsymbol{w}\|^2_{L^2(\Omega_T)} + \tfrac{1}{\lambda} \, \|\boldsymbol{z}\|^2_{L^2(\Omega_T)} \,, \tag{6.4.24}$$

$$\int_0^T |\boldsymbol{w}|_1 \, |\boldsymbol{z}|_1 \, \mathrm{d}t \;\leq \tfrac{\lambda}{4} \, \|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_T)} + \tfrac{1}{\lambda} \, \|\nabla \boldsymbol{z}\|^2_{L^2(\Omega_T)} \,. \tag{6.4.25}$$

Using (6.4.12), we find that

$$2 \int_0^T \|\boldsymbol{w}\|_{0,\infty} \, |z_2|_1 \, |\boldsymbol{w}|_1 \, \mathrm{d}t \;\leq C \int_0^T \|\boldsymbol{w}\|_{0,\infty} \, |\boldsymbol{w}|_1 \, \mathrm{d}t \,. \tag{6.4.26}$$

Putting (6.4.24)-(6.4.26) in (6.4.23) leads to

$$\begin{aligned} &\tfrac{1}{2} \left(\|\boldsymbol{w}(T)\|_0^2 + \tfrac{1}{\lambda} \, \|\boldsymbol{z}(T)\|_0^2 \right) + (\rho - \tfrac{\lambda}{4}) \, \|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_T)} \\ &\qquad \leq \tfrac{\mu^2}{4\,\lambda} \, \|\boldsymbol{w}\|^2_{L^2(\Omega_T)} + C \int_0^T \|\boldsymbol{w}\|_{0,\infty} \, |\boldsymbol{w}|_1 \, \mathrm{d}t \,. \end{aligned} \tag{6.4.27}$$

To deal with the integral in the right hand side of (6.4.27), we first note that the Sobolev interpolation result (2.1.1), in one space dimension, gives

$$\begin{aligned} \|\boldsymbol{w}\|_{0,\infty} &\leq C \, \|\boldsymbol{w}\|_0^{\frac{1}{2}} \, \|\boldsymbol{w}\|_1^{\frac{1}{2}} \\ &\leq C \, \|\boldsymbol{w}\|_0^{\frac{1}{2}} \, \left(\|\boldsymbol{w}\|_0^2 + |\boldsymbol{w}|_1^2 \right)^{\frac{1}{4}} \\ &\leq C \, \left(\|\boldsymbol{w}\|_0 + \|\boldsymbol{w}\|_0^{\frac{1}{2}} \, |\boldsymbol{w}|_1^{\frac{1}{2}} \right) . \end{aligned} \tag{6.4.28}$$

Therefore, we have from (6.4.28) and the Young's inequality for any $\delta > 0$ that

$$\begin{aligned} C \int_0^T \|\boldsymbol{w}\|_{0,\infty} \, |\boldsymbol{w}|_1 \, \mathrm{d}t \;&\leq C \int_0^T \left(\|\boldsymbol{w}\|_0 \, |\boldsymbol{w}|_1 + \|\boldsymbol{w}\|_0^{\frac{1}{2}} \, |\boldsymbol{w}|_1^{\frac{3}{2}} \right) \, \mathrm{d}t \\ &\leq \int_0^T \left(C(\delta) \, \|\boldsymbol{w}\|_0^2 + \delta \, |\boldsymbol{w}|_1^2 \right) \, \mathrm{d}t \\ &= C(\delta) \, \|\boldsymbol{w}\|^2_{L^2(\Omega_T)} + \delta \, \|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_T)} \,. \end{aligned} \tag{6.4.29}$$

As $\lambda < 4\,\rho$, we choose $\delta = \rho - \frac{\lambda}{4}$ in (6.4.29) and then we combine the resulting inequality with (6.4.27) to infer

$$\|\boldsymbol{w}(T)\|_0^2 + \tfrac{1}{\lambda} \, \|\boldsymbol{z}(T)\|_0^2 \leq C \, \|\boldsymbol{w}\|^2_{L^2(\Omega_T)} \,. \tag{6.4.30}$$

Applying the integral version of the Grönwall lemma, (2.1.5), leads to

$$\|\boldsymbol{w}(T)\|_0^2 + \tfrac{1}{\lambda} \, \|\boldsymbol{z}(T)\|_0^2 \leq 0 \,.$$

Thus, we conclude $w_1 \equiv w_2$ and $z_1 \equiv z_2$ as required. □

6.5 An error estimate

In this section we study the error estimate between the weak solution of (Q) and their fully discrete approximations defined by (6.3.6a)-(6.3.6b). Additionally to the uniqueness requirements, the derivation of an error estimate requires extra regularity on the time derivatives of the approximate solutions. The details are given in the following theorem.

Theorem 6.5.1 Let all the assumptions of Theorem 6.4.2 hold. If $\lambda < 4\,\rho$ and

$$\|\tfrac{\partial W_\varepsilon}{\partial t}\|_{L^2(\Omega_T)} + \|\tfrac{\partial Z_\varepsilon}{\partial t}\|_{L^2(\Omega_T)} + \|Z_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq C\,, \tag{6.5.1}$$

then the solution $\{W_\varepsilon, Z_\varepsilon\}$ of $(\mathrm{Q}_\varepsilon^{h,\Delta t})$, $h\,,\,\Delta t\,\leq\,1$, satisfies the following error bound[1]:

$$\begin{aligned}
&\|w - W_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z - Z_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\qquad \leq C\,\left(h + \Delta t + \varepsilon^2 + \|\nabla(I - \pi^h)w\|_{L^2(\Omega_T)} + \|\nabla(I-\pi^h)z\|_{L^2(\Omega_T)}\right). \qquad (6.5.2)
\end{aligned}$$

Furthermore, if $w\,,\,z \in L^2(0,T;H^2(\Omega))$ then

$$\|w - W_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z - Z_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C\,\left(h + \Delta t + \varepsilon^2\right). \tag{6.5.3}$$

Proof: We first mention that $\pi^h w$ and $\pi^h z$ are well defined since $w(\cdot,t)\,,\,z(\cdot,t) \in H^1(\Omega)$ for *a.e.* $t \in (0,T)$ and the Sobolev embedding result $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$ holds in one space dimension. Noting this, we set

$$\begin{aligned}
e_y^A &:= y - \pi^h y\,, \\
e_{y,\varepsilon}^{(\pm)} &:= y - Y_\varepsilon^{(\pm)}, \qquad (6.5.4)\\
E_{y,\varepsilon}^{(\pm)} &:= \pi^h y - Y_\varepsilon^{(\pm)},
\end{aligned}$$

where $y \equiv w$ and z , $Y_\varepsilon^{(\pm)} \equiv W_\varepsilon^{(\pm)}$ and $Z_\varepsilon^{(\pm)}$, respectively.

[1] If (6.5.1) holds then, using classical compactness arguments, we have that $\frac{\partial w}{\partial t}\,,\,\frac{\partial z}{\partial t}\ \in L^2(\Omega_T)$ and $z \in L^\infty(0,T;H^1(\Omega))$. In addition, the solution $\{w,z\}$ will be unique under the assumption $\lambda < 4\,\rho$; see Theorem 6.4.2.

On subtracting (6.4.1a,b) from (6.4.11a,b) respectively, it follows for *a.e.* $t \in (0,T)$ and for all $\chi \in S^h$ that

$$\left(\tfrac{\partial e_{w,\varepsilon}}{\partial t},\chi\right) + \rho\,\left(\nabla e^+_{w,\varepsilon},\nabla\chi\right) = \lambda\,\left(\mathcal{V}(w)\nabla z,\nabla\chi\right) - \lambda\,\left(\Pi_\varepsilon(W^+_\varepsilon)\nabla Z^+_\varepsilon,\nabla\chi\right) + \left\{\left(\tfrac{\partial W_\varepsilon}{\partial t},\chi\right)^h - \left(\tfrac{\partial W_\varepsilon}{\partial t},\chi\right)\right\}, \tag{6.5.5a}$$

$$\begin{aligned}\left(\tfrac{\partial e_{z,\varepsilon}}{\partial t},\chi\right) + \left(e^+_{z,\varepsilon},\chi\right) + \left(\nabla e^+_{z,\varepsilon},\nabla\chi\right) + \lambda\,\left(\nabla e^+_{w,\varepsilon},\nabla\chi\right) &= \left\{\left(\tfrac{\partial Z_\varepsilon}{\partial t},\chi\right)^h - \left(\tfrac{\partial Z_\varepsilon}{\partial t},\chi\right)\right\}\\ &+ \left\{\left(Z^+_\varepsilon,\chi\right)^h - \left(Z^+_\varepsilon,\chi\right)\right\} + \mu\,\theta\,\left(e^+_{w,\varepsilon},\chi\right) + \mu\,(1-\theta)\,\left(e^-_{w,\varepsilon},\chi\right)\\ &+ \mu\,\theta\,\left\{\left(W^+_\varepsilon,\chi\right) - \left(W^+_\varepsilon,\chi\right)^h\right\} + \mu\,(1-\theta)\,\left\{\left(W^-_\varepsilon,\chi\right) - \left(W^-_\varepsilon,\chi\right)^h\right\}.\end{aligned} \tag{6.5.5b}$$

Hence, choosing $\chi \equiv E^+_{w,\varepsilon} \in S^h$ in (6.5.5a) and $\chi \equiv \frac{1}{\lambda}\,E^+_{z,\varepsilon} \in S^h$ in (6.5.5b) and summing the resulting equations yields that

$$\begin{aligned}&\tfrac{1}{2}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\,\|e_{w,\varepsilon}\|_0^2 + \tfrac{1}{2\,\lambda}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\,\|e_{z,\varepsilon}\|_0^2 + \tfrac{1}{\lambda}\,\|e^+_{z,\varepsilon}\|_0^2 + \rho\,|e^+_{w,\varepsilon}|_1^2 + \tfrac{1}{\lambda}\,|e^+_{z,\varepsilon}|_1^2\\
&= \left[\left(\tfrac{\partial e_{w,\varepsilon}}{\partial t},e^A_w\right) + \tfrac{1}{\lambda}\,\left(\tfrac{\partial e_{z,\varepsilon}}{\partial t},e^A_z\right)\right]\\
&+ \left[\left(\tfrac{\partial e_{w,\varepsilon}}{\partial t},W^+_\varepsilon - W_\varepsilon\right) + \tfrac{1}{\lambda}\,\left(\tfrac{\partial e_{z,\varepsilon}}{\partial t},Z^+_\varepsilon - Z_\varepsilon\right)\right]\\
&+ \left[\,\rho\,\left(\nabla e^+_{w,\varepsilon},\nabla e^A_w\right) + \tfrac{1}{\lambda}\,\left(\nabla e^+_{z,\varepsilon},\nabla e^A_z\right) + \tfrac{1}{\lambda}\,\left(e^+_{z,\varepsilon},e^A_z\right)\right]\\
&+ \Bigg[\left\{\left(\tfrac{\partial W_\varepsilon}{\partial t},E^+_{w,\varepsilon}\right)^h - \left(\tfrac{\partial W_\varepsilon}{\partial t},E^+_{w,\varepsilon}\right)\right\} + \tfrac{1}{\lambda}\,\left\{\left(\tfrac{\partial Z_\varepsilon}{\partial t},E^+_{z,\varepsilon}\right)^h - \left(\tfrac{\partial Z_\varepsilon}{\partial t},E^+_{z,\varepsilon}\right)\right\}\\
&\qquad + \tfrac{1}{\lambda}\,\left\{\left(Z^+_\varepsilon,E^+_{z,\varepsilon}\right)^h - \left(Z^+_\varepsilon,E^+_{z,\varepsilon}\right)\right\}\Bigg]\\
&+ \left[\left(\nabla e^+_{w,\varepsilon},\nabla e^A_z\right) - \lambda\,\left(\mathcal{V}(w)\nabla e^+_{z,\varepsilon},\nabla e^A_w\right)\right]\\
&+ \left[\left([\lambda\,\mathcal{V}(w) - 1]\,\nabla e^+_{z,\varepsilon},\nabla e^+_{w,\varepsilon}\right)\right]\\
&+ \left[\,\lambda\,\left(\left[\mathcal{V}(w) - \Pi_\varepsilon(W^+_\varepsilon)\right]\nabla Z^+_\varepsilon,\nabla E^+_{w,\varepsilon}\right)\right]\\
&+ \left[\,\tfrac{\mu\,\theta}{\lambda}\,\left(e^+_{w,\varepsilon},E^+_{z,\varepsilon}\right) + \tfrac{\mu\,(1-\theta)}{\lambda}\,\left(e^-_{w,\varepsilon},E^+_{z,\varepsilon}\right)\right]\\
&+ \left[\,\tfrac{\mu\,\theta}{\lambda}\,\left\{\left(W^+_\varepsilon,E^+_{z,\varepsilon}\right) - \left(W^+_\varepsilon,E^+_{z,\varepsilon}\right)^h\right\} + \tfrac{\mu\,(1-\theta)}{\lambda}\,\left\{\left(W^-_\varepsilon,E^+_{z,\varepsilon}\right) - \left(W^-_\varepsilon,E^+_{z,\varepsilon}\right)^h\right\}\right]\\
&:= \sum_{i=1}^{9}\,[I_i]\,,\end{aligned} \tag{6.5.6}$$

where we have noticed from (6.5.4) that

$$E^{(\pm)}_{y,\varepsilon} = e^{(\pm)}_{y,\varepsilon} - e^A_y = e_{y,\varepsilon} - e^A_y + (Y_\varepsilon - Y^{(\pm)}_\varepsilon)\,.$$

We now bound each term on the right hand side of (6.5.6) separately.

Using the Cauchy-Schwarz inequality gives that

$$I_1 \leq C\, \left(\|\tfrac{\partial e_{w,\varepsilon}}{\partial t}\|_0\, \|e_w^A\|_0 + \|\tfrac{\partial e_{z,\varepsilon}}{\partial t}\|_0\, \|e_z^A\|_0\right) := \tilde{I}_1\,, \tag{6.5.7}$$

$$I_2 \leq C\, \left(\|\tfrac{\partial e_{w,\varepsilon}}{\partial t}\|_0\, \|W_\varepsilon^+ - W_\varepsilon\|_0 + \|\tfrac{\partial e_{z,\varepsilon}}{\partial t}\|_0\, \|Z_\varepsilon^+ - Z_\varepsilon\|_0\right) := \tilde{I}_2\,, \tag{6.5.8}$$

$$I_3 \leq C\, \left(|e_{w,\varepsilon}^+|_1\, |e_w^A|_1 + |e_{z,\varepsilon}^+|_1\, |e_z^A|_1 + \|e_{z,\varepsilon}^+\|_0\, \|e_z^A\|_0\right) := \tilde{I}_3\,. \tag{6.5.9}$$

With the aid of (2.4.19), we have that

$$I_4 \leq C\, h\, \left(\|\tfrac{\partial W_\varepsilon}{\partial t}\|_0\, |E_{w,\varepsilon}^+|_1 + \|\tfrac{\partial Z_\varepsilon}{\partial t}\|_0\, |E_{z,\varepsilon}^+|_1 + \|Z_\varepsilon^+\|_0\, |E_{z,\varepsilon}^+|_1\right) := \tilde{I}_4\,. \tag{6.5.10}$$

Noting the Cauchy-Schwarz inequality, (6.1.2) and (6.4.2c) leads to

$$I_5 \leq |e_{w,\varepsilon}^+|_1\, |e_z^A|_1 + |e_{z,\varepsilon}^+|_1\, |e_w^A|_1 := \tilde{I}_5\,. \tag{6.5.11}$$

We also obtain from (6.4.2c), (6.1.2) and the Young's inequality that

$$I_6 \leq |e_{w,\varepsilon}^+|_1\, |e_{z,\varepsilon}^+|_1 \leq \tfrac{\lambda}{4}\, |e_{w,\varepsilon}^+|_1^2 + \tfrac{1}{\lambda}\, |e_{z,\varepsilon}^+|_1^2\,. \tag{6.5.12}$$

It follows from the Hölder's inequality, the last bound in (6.5.1), (6.3.5), (2.4.15), the Lipschitz continuity of $\mathcal{V}_\varepsilon$, (6.2.6) and (6.5.4) that

$$\begin{aligned}
I_7 &\leq \lambda\, |Z_\varepsilon^+|_1\, \|\Pi_\varepsilon(W_\varepsilon^+) - \mathcal{V}(w)\|_{0,\infty}\, |E_{w,\varepsilon}^+|_1 \\
&\leq C\, \|\Pi_\varepsilon(W_\varepsilon^+) - \mathcal{V}(w)\|_{0,\infty}\, |E_{w,\varepsilon}^+|_1 \\
&\leq C\, \left(\|\Pi_\varepsilon(W_\varepsilon^+) - \mathcal{V}_\varepsilon(W_\varepsilon^+)\|_{0,\infty} + \|\mathcal{V}_\varepsilon(W_\varepsilon^+) - \mathcal{V}_\varepsilon(w)\|_{0,\infty} + \|\mathcal{V}_\varepsilon(w) - \mathcal{V}(w)\|_{0,\infty}\right) |E_{w,\varepsilon}^+|_1 \\
&\leq C\, \left(h^{\frac{1}{2}}\, |W_\varepsilon^+|_1 + \|e_{w,\varepsilon}^+\|_{0,\infty} + \varepsilon\right) |E_{w,\varepsilon}^+|_1 \\
&\leq C\, \left(h^{\frac{1}{2}}\, \|W_\varepsilon^+\|_1 + \|e_{w,\varepsilon}^+\|_{0,\infty} + \varepsilon\right) \left(|e_{w,\varepsilon}^+|_1 + |e_w^A|_1\right) \\
&:= I_{7,1} + I_{7,2} + I_{7,3}\,,
\end{aligned} \tag{6.5.13}$$

where

$$\begin{aligned}
I_{7,1} &:= C\, \left(h^{\frac{1}{2}}\, \|W_\varepsilon^+\|_1 + \varepsilon\right)\, |e_{w,\varepsilon}^+|_1\,, \\
I_{7,2} &:= C\, \|e_{w,\varepsilon}^+\|_{0,\infty}\, |e_{w,\varepsilon}^+|_1\,, \\
I_{7,3} &:= C\, \left(h^{\frac{1}{2}}\, \|W_\varepsilon^+\|_1 + \|e_{w,\varepsilon}^+\|_{0,\infty} + \varepsilon\right)\, |e_w^A|_1\,.
\end{aligned}$$

But, the Young's inequality gives, on noting the assumption $\lambda < 4\,\rho$, that

$$I_{7,1} \leq C\,\left(\,h\,\|W_\varepsilon^+\|_1^2 + \varepsilon^2\,\right) + \tfrac{4\,\rho-\lambda}{8}\,|e_{w,\varepsilon}^+|_1^2\,. \tag{6.5.14}$$

Similarly to (6.4.28), we obtain from (2.1.1) and the Young's inequality that

$$\begin{aligned} I_{7,2} &= C\,\|e_{w,\varepsilon}^+\|_{0,\infty}\,|e_{w,\varepsilon}^+|_1 \\ &\leq C\,\|e_{w,\varepsilon}^+\|_0^{\frac{1}{2}}\,\|e_{w,\varepsilon}^+\|_1^{\frac{1}{2}}\,|e_{w,\varepsilon}^+|_1 \\ &\leq C\,\left(\|e_{w,\varepsilon}^+\|_0\,|e_{w,\varepsilon}^+|_1 + \|e_{w,\varepsilon}^+\|_0^{\frac{1}{2}}\,|e_{w,\varepsilon}^+|_1^{\frac{3}{2}}\right) \\ &\leq C\,\|e_{w,\varepsilon}^+\|_0^2 + \tfrac{4\,\rho-\lambda}{8}\,|e_{w,\varepsilon}^+|_1^2 \\ &\leq C\,\|e_{w,\varepsilon}\|_0^2 + C\,\|W_\varepsilon^+ - W_\varepsilon\|_0^2 + \tfrac{4\,\rho-\lambda}{8}\,|e_{w,\varepsilon}^+|_1^2\,. \end{aligned} \tag{6.5.15}$$

Noting the Cauchy-Schwarz inequality and the Young's inequality leads to

$$\begin{aligned} I_8 &\leq \tfrac{\mu\,\theta}{\lambda}\,\|e_{w,\varepsilon}^+\|_0\,\|E_{z,\varepsilon}^+\|_0 + \tfrac{\mu\,(1-\theta)}{\lambda}\,\|e_{w,\varepsilon}^-\|_0\,\|E_{z,\varepsilon}^+\|_0 \\ &\leq C\,\|e_{w,\varepsilon}^+\|_0^2 + C\,\|e_{w,\varepsilon}^-\|_0^2 + C\,\|E_{z,\varepsilon}^+\|_0^2 \\ &\leq C\,\|e_{w,\varepsilon}\|_0^2 + C\,\|e_{z,\varepsilon}\|_0^2 + C\,\left(\|W_\varepsilon^+ - W_\varepsilon\|_0^2 + \|W_\varepsilon^- - W_\varepsilon\|_0^2 + \|Z_\varepsilon^+ - Z_\varepsilon\|_0^2 + \|e_z^A\|_0^2\right)\,. \end{aligned} \tag{6.5.16}$$

Finally, we use (2.4.19) and the Young's inequality to obtain that

$$\begin{aligned} I_9 &\leq C\,h\,\left(|W_\varepsilon^+|_1 + |W_\varepsilon^-|_1\right)\,\|E_{z,\varepsilon}^+\|_0 \\ &\leq C\,h\,\left(\|W_\varepsilon^+\|_1 + \|W_\varepsilon^-\|_1\right)\,\|E_{z,\varepsilon}^+\|_0 := \tilde{I}_9\,. \end{aligned} \tag{6.5.17}$$

Now, combining (6.5.6)→(6.5.17) yields that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\,\left(\|e_{w,\varepsilon}\|_0^2 + \tfrac{1}{\lambda}\,\|e_{z,\varepsilon}\|_0^2\right) \leq C\,\left(\|e_{w,\varepsilon}\|_0^2 + \tfrac{1}{\lambda}\,\|e_{z,\varepsilon}\|_0^2\right) + \sum_{i=1}^{9}\tilde{I}_i\,, \tag{6.5.18}$$

where

$$\begin{aligned} \tilde{I}_6 &:= 0 \\ \tilde{I}_7 &:= I_{7,3} + C\,\left(\,h\,\|W_\varepsilon^+\|_1^2 + \varepsilon^2 + \|W_\varepsilon^+ - W_\varepsilon\|_0^2\right)\,, \\ \tilde{I}_8 &:= C\,\left(\|W_\varepsilon^+ - W_\varepsilon\|_0^2 + \|W_\varepsilon^- - W_\varepsilon\|_0^2 + \|Z_\varepsilon^+ - Z_\varepsilon\|_0^2 + \|e_z^A\|_0^2\right)\,. \end{aligned}$$

Applying the Grönwall lemma to (6.5.18) leads to for *a.e.* $t \in (0,T)$

$$\|e_{w,\varepsilon}(t)\|_0^2 + \tfrac{1}{\lambda}\,\|e_{z,\varepsilon}(t)\|_0^2 \le e^{CT}\,\left(\|e_{w,\varepsilon}(0)\|_0^2 + \tfrac{1}{\lambda}\,\|e_{z,\varepsilon}(0)\|_0^2\right) + e^{CT}\int_0^T \sum_{i=1}^{9} \tilde{I}_i\,\mathrm{d}t\,. \tag{6.5.19}$$

To bound the right hand side of (6.5.19), we first note from (6.4.2b), the assumption $w^0\,,\, z^0 \in H^1(\Omega)$ (see Theorem 6.4.1), (3.1.3) and (2.4.16) that

$$\begin{aligned}\|e_{w,\varepsilon}(0)\|_0^2 &= \|w^0 - W_\varepsilon^0\|_0^2 \le C\,h^2|w^0|_1^2 \le C\,h^2,\\ \|e_{z,\varepsilon}(0)\|_0^2 &= \|z^0 - Z_\varepsilon^0\|_0^2 \le C\,h^2|z^0|_1^2 \le C\,h^2.\end{aligned} \tag{6.5.20}$$

We also use the estimate (2.4.16) to find that

$$\begin{aligned}\|e_w^A\|_0^2 &= \|(I-\pi^h)w\|_0^2 \le C\,h^2|w|_1^2\,,\\ \|e_z^A\|_0^2 &= \|(I-\pi^h)z\|_0^2 \le C\,h^2|z|_1^2\,.\end{aligned} \tag{6.5.21}$$

Similarly to (6.4.7), we have from (6.5.1) that

$$\begin{aligned}&\|W_\varepsilon^{\pm} - W_\varepsilon\|_{L^2(\Omega_T)}^2 + \|Z_\varepsilon^{\pm} - Z_\varepsilon\|_{L^2(\Omega_T)}^2\\ &\qquad\le (\Delta t)^2\,\|\tfrac{\partial W_\varepsilon}{\partial t}\|_{L^2(\Omega_T)}^2 + (\Delta t)^2\,\|\tfrac{\partial Z_\varepsilon}{\partial t}\|_{L^2(\Omega_T)}^2 \le C\,(\Delta t)^2\,.\end{aligned} \tag{6.5.22}$$

On noting (6.5.4), (6.4.2a), (6.4.6a,b) and (2.4.16), we deduce that

$$\begin{aligned}&\|E_{w,\varepsilon}^+\|_{L^2(0,T;H^1(\Omega))} + \|E_{z,\varepsilon}^+\|_{L^2(0,T;H^1(\Omega))}\\ &\quad\le \|e_{w,\varepsilon}^+\|_{L^2(0,T;H^1(\Omega))} + \|e_{z,\varepsilon}^+\|_{L^2(0,T;H^1(\Omega))} + \|e_w^A\|_{L^2(0,T;H^1(\Omega))} + \|e_z^A\|_{L^2(0,T;H^1(\Omega))}\\ &\quad\le C\,.\end{aligned} \tag{6.5.23}$$

Now, using the Hölder's inequality, (6.4.2a), (6.4.6a,b), (6.5.1), (6.5.21), (6.5.22) and (6.5.23), we can obtain the following estimates:

$$\begin{aligned}\int_0^T \tilde{I}_1\,\mathrm{d}t\ &\le C\,h\,\left(\|\tfrac{\partial e_{w,\varepsilon}}{\partial t}\|_{L^2(\Omega_T)}\,\|w\|_{L^2(0,T;H^1(\Omega))} + \|\tfrac{\partial e_{z,\varepsilon}}{\partial t}\|_{L^2(\Omega_T)}\,\|z\|_{L^2(0,T;H^1(\Omega))}\right)\\ &\le C\,h\,.\end{aligned} \tag{6.5.24a}$$

$$\begin{aligned}\int_0^T \tilde{I}_2\,\mathrm{d}t\ &\le C\,\left(\|\tfrac{\partial e_{w,\varepsilon}}{\partial t}\|_{L^2(\Omega_T)}\|W_\varepsilon^+ - W_\varepsilon\|_{L^2(\Omega_T)} + \|\tfrac{\partial e_{z,\varepsilon}}{\partial t}\|_{L^2(\Omega_T)}\|Z_\varepsilon^+ - Z_\varepsilon\|_{L^2(\Omega_T)}\right)\\ &\le C\,\Delta t\,.\end{aligned} \tag{6.5.24b}$$

$$\begin{aligned}\int_0^T \tilde{I}_3 \ \mathrm{d}t \ &\le C\ \left(\|e^+_{w,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}\|\nabla e^A_w\|_{L^2(\Omega_T)} + \|e^+_{z,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}\|\nabla e^A_z\|_{L^2(\Omega_T)}\right)\\ &\quad + C\,h\,\|e^+_{z,\varepsilon}\|_{L^2(\Omega_T)}\,\|z\|_{L^2(0,T;H^1(\Omega))}\\ &\le C\ \left(h + \|\nabla e^A_w\|_{L^2(\Omega_T)} + \|\nabla e^A_z\|_{L^2(\Omega_T)}\right)\ . \end{aligned} \tag{6.5.24c}$$

$$\begin{aligned}\int_0^T \tilde{I}_4 \ \mathrm{d}t \ &\le C\,h\,\|\tfrac{\partial W_\varepsilon}{\partial t}\|_{L^2(\Omega_T)}\|E^+_{w,\varepsilon}\|_{L^2(0,T;H^1(\Omega))} + C\,h\,\|\tfrac{\partial Z_\varepsilon}{\partial t}\|_{L^2(\Omega_T)}\|E^+_{z,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}\\ &\quad + C\,h\,\|Z^+_\varepsilon\|_{L^2(\Omega_T)}\|E^+_{z,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}\\ &\le C\,h\,. \end{aligned} \tag{6.5.24d}$$

$$\begin{aligned}\int_0^T \tilde{I}_5 \ \mathrm{d}t \ &\le \|e^+_{w,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}\|\nabla e^A_z\|_{L^2(\Omega_T)} + \|e^+_{z,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}\|\nabla e^A_w\|_{L^2(\Omega_T)}\\ &\le C\ \left(\|\nabla e^A_z\|_{L^2(\Omega_T)} + \|\nabla e^A_w\|_{L^2(\Omega_T)}\right)\ . \end{aligned} \tag{6.5.24e}$$

$$\begin{aligned}\int_0^T \tilde{I}_7 \ \mathrm{d}t \ &\le C\ \left(h^{\frac{1}{2}}\|W^+_\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|e^+_{w,\varepsilon}\|_{L^2(0,T;L^\infty(\Omega))} + \varepsilon\right)\|\nabla e^A_w\|_{L^2(\Omega_T)}\\ &\quad + C\ \left(h\,\|W^+_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))} + \varepsilon^2 + \|W^+_\varepsilon - W_\varepsilon\|^2_{L^2(\Omega_T)}\right)\\ &\le C\ \left(h + \varepsilon^2 + (\Delta t)^2 + \|\nabla e^A_w\|_{L^2(\Omega_T)}\right)\ . \end{aligned} \tag{6.5.24f}$$

$$\begin{aligned}\int_0^T \tilde{I}_8 \ \mathrm{d}t \ &\le C\ \left(\|W^+_\varepsilon - W_\varepsilon\|^2_{L^2(\Omega_T)} + \|W^-_\varepsilon - W_\varepsilon\|^2_{L^2(\Omega_T)} + \|Z^+_\varepsilon - Z_\varepsilon\|^2_{L^2(\Omega_T)}\right)\\ &\quad + C\,h^2\,\|z\|^2_{L^2(0,T;H^1(\Omega))}\\ &\le C\ \left((\Delta t)^2 + h^2\right)\ . \end{aligned} \tag{6.5.24g}$$

$$\begin{aligned}\int_0^T \tilde{I}_9 \ \mathrm{d}t \ &\le C\,h\ \left(\|W^+_\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|W^-_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\right)\|E^+_{z,\varepsilon}\|_{L^2(\Omega_T)}\\ &\le C\,h\,. \end{aligned} \tag{6.5.24h}$$

Combining (6.5.19), (6.5.20) and (6.5.24a)-(6.5.24h) yields for $h\,,\ \Delta t \le 1$ and for $a.e.\ t \in (0,T)$ that

$$\begin{aligned}\|e_{w,\varepsilon}(t)\|^2_0 + \|e_{z,\varepsilon}(t)\|^2_0 &\le C\ \left(h^2 + (\Delta t)^2 + \varepsilon^2 + h + \Delta t + \|\nabla e^A_w\|_{L^2(\Omega_T)} + \|\nabla e^A_z\|_{L^2(\Omega_T)}\right)\\ &\le C\ \left(h + \Delta t + \varepsilon^2 + \|\nabla e^A_w\|_{L^2(\Omega_T)} + \|\nabla e^A_z\|_{L^2(\Omega_T)}\right)\,.\end{aligned}$$

This gives the estimate (6.5.2) .

If $w\,,\,z \in L^2(0,T;H^2(\Omega))$, the result (6.5.3) follows immediately from (6.5.2) on noting the following estimate (see Theorem 3.1.6 in Ciarlet [23]):

$$|(I-\pi^h)\eta|_1 \leq C\,h\,|\eta|_2 \qquad \forall \eta \in H^2(\Omega)\,.$$

□

At the end of the proof, we comment that some terms of $\sum_{i=1}^{9} \tilde{I}_i$ can be treated differently to obtain $\mathcal{O}(h^2)$ bound instead of $\mathcal{O}(h)$ bound. However, this does not give an improvement of the overall bound as the bound in (6.5.24f) will still have the order $\mathcal{O}(h)$. For instance, we were unable to obtain a better bound for the term $\|\Pi_\varepsilon(W_\varepsilon^+) - \mathcal{V}_\varepsilon(W_\varepsilon^+)\|_{0,\infty}$ in (6.5.13) where we have applied (6.3.5) followed by an inverse inequality.

Remark 6.5.1 If we replace the assumption $\lambda < 4\,\rho$ in Theorem 6.5.1 by

$$\frac{\lambda}{1-\lambda\,\delta_2} < 4\,(\rho-\delta_1)\,,$$

where δ_1 and δ_2 are fixed positive constants such that $\delta_1 < \rho$ and $\delta_2 < \frac{1}{\lambda}$, then we can repeat the argument presented in Theorem 6.5.1 to show the following error bound:

$$\begin{aligned}
&\|w - W_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z - Z_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\qquad\qquad + \|w - W_\varepsilon^+\|^2_{L^2(0,T;H^1(\Omega))} + \|z - Z_\varepsilon^+\|^2_{L^2(0,T;H^1(\Omega))} \\
&\quad \leq C(\delta_1^{-1},\delta_2^{-1})\,\left(h + \Delta t + \varepsilon^2 + \|\nabla(I-\pi^h)w\|_{L^2(\Omega_T)} + \|\nabla(I-\pi^h)z\|_{L^2(\Omega_T)}\right),
\end{aligned}$$

which is of order $\mathcal{O}(h + \Delta t + \varepsilon^2)$ if $w\,,\,z \in L^2(0,T;H^2(\Omega))$.

6.6 Long time behaviour

This section is devoted to an investigation of the long time behaviour of the solutions of (Q). Aranson *et al.* [4] have shown from linear stability theory that the condition $\mu > \rho$ is necessary to have size segregation. Thereafter, Galiano *et al.* [34] have shown that the condition $\mu > \rho$ need not to be sufficient. Namely, they showed that for parameter values of μ, ρ and λ chosen such that $\mu > \rho$ and

$$\rho\,\lambda > \frac{\mu^2\,L^4}{8\,(L^2+1)}\,, \tag{6.6.1}$$

the solutions, w and z, converge as time tends to infinity to the constant steady-state solutions given, respectively, by

$$\bar{w} = \frac{1}{L}\int_0^L w^0(x)\;\mathrm{d}x \quad \text{and} \quad \bar{z} = \frac{\mu}{L}\int_0^L w^0(x)\;\mathrm{d}x\,.$$

We actually believe that the analysis presented by Galiano *et al.* [34], is not sufficient to conclude that the solutions converges to constant steady-state solutions. In particular, we believe that the application of the Grönwall lemma that has been used by Galiano *et al.* in the last step of the proof of Theorem 1.3, in their work in [34], is wrong. We see counterexamples later on in this section.

In what follows, we revisit the results of Galiano *et al.* [34] (Section 4). We modify the proof of Theorem 1.3, in [34], to obtain an estimate that in some sense explains the long time behaviour of the solutions of (Q). Alternatively to the approach in [34], our analysis relies on similar techniques to that shown previously in Section 4.4 where we exploit the regularization introduced in Section 6.2. We also see that the use of an optimal Poincaré inequality results in an improved condition to (6.6.1). At the end of this section, we comment on the wrong use of the Grönwall lemma in [34] by provision of a counterexample.

First, we note from the Poincaré inequality that for all $\eta \in H^1(\Omega)$ such that $\fint \eta = 0$ we have

$$\|\eta\|_0^2 \leq C_p\,|\eta|_1^2\,.$$

For the analysis that follows, it is important to be as precise as possible about the constant "C_p" in the above inequality. With this in mind we note the following lemma:

Lemma 6.6.1 Let $\Omega := (0, L) \subset \mathbb{R}$. Then

$$\|u\|_0^2 \leq \tfrac{L^2}{\pi^2}\, |u|_1^2\,, \tag{6.6.2}$$

for all $u \in H^1(\Omega)$ satisfying $\displaystyle\int_0^L u(x)\ \mathrm{d}x\ = 0$.

Proof: See the proof of Lemma 2.1 in Bebendorf [14]. □

Theorem 6.6.2 Let $\rho\,,\ \lambda > 0$ and $\mu \geq 0$ and let $w^0,\ z^0\ \in H^1(\Omega)$ with $|w^0(\cdot)| \leq 1$ *a.e.* in Ω. Then for any $T > 0$ there exists a weak solution $\{w, z\}$ of (1.2.1a)-(1.2.1d) satisfying (6.4.2a)-(6.4.2c), (6.4.11a,b) and that $\fint\ w\ = \fint\ w^0$ for *a.e.* $t \in (0, T)$. Furthermore, if $|w^0| \leq l$ in Ω for some $l < 1$, $\mu \fint\ w^0 = \fint\ z^0$ and

$$\rho\,\lambda > \frac{\mu^2\, L^4}{4\,\pi^2\,(L^2 + \pi^2)}\,, \tag{6.6.3}$$

then there exist $C_0(w^0, z^0) \geq 0$ and $\delta_1\,,\ \delta_2 > 0$, depend on the parameters $\rho\,,\ \lambda\,,\ \mu$ and L, such that

$$\begin{aligned}
\int_0^T \|w - \textstyle\fint\ w^0\|_0^2\ \mathrm{d}t \quad &\leq \tfrac{C_0}{\lambda\,\delta_1}\ \left(1 - e^{-\delta_1\,T}\right)\,,\\
\int_0^T \|z - \textstyle\fint\ z^0\|_0^2\ \ \mathrm{d}t \quad &\leq \tfrac{C_0}{\delta_2}\ \left(1 - e^{-\delta_2\,T}\right)\,.
\end{aligned} \tag{6.6.4}$$

Proof: The existence follows from Theorem 6.4.2. As $\mu \fint\ w^0\ = \fint\ z^0$, we easily obtain from (6.4.11a,b) that for *a.e.* $t \in (0, T)$

$$\mu \fint\ w(t) = \mu \fint\ w^0 = \fint\ z^0 = \fint\ z(t)\,. \tag{6.6.5}$$

To show the result (6.6.4), we need to introduce a simple modification on the regularized function $\Phi_\varepsilon(s)$. On noting the assumption on w^0, we define for $\varepsilon \in (0, 1 - l)$ the twice continuously differentiable function $\widetilde{\Phi}_\varepsilon : [-1, 1] \to \mathbb{R}$ given by

$$\widetilde{\Phi}_\varepsilon(s) := \begin{cases}
\frac{\lambda}{2}\left[(1+s)\ln(\frac{1+s}{1+\fint\ w^0}) + \frac{1}{2\varepsilon}(1-s)^2 + (1-s)\ln(\frac{\varepsilon}{1-\fint\ w^0}) - \frac{\varepsilon}{2}\right] & \text{if } 1 - \varepsilon \leq s \leq 1\,,\\[2mm]
\frac{\lambda}{2}\left[(1+s)\ln(\frac{1+s}{1+\fint\ w^0}) + (1-s)\ln(\frac{1-s}{1-\fint\ w^0})\right] & \text{if } |s| \leq 1 - \varepsilon\,,\\[2mm]
\frac{\lambda}{2}\left[(1-s)\ln(\frac{1-s}{1-\fint\ w^0}) + \frac{1}{2\varepsilon}(1+s)^2 + (1+s)\ln(\frac{\varepsilon}{1+\fint\ w^0}) - \frac{\varepsilon}{2}\right] & \text{if } -1 \leq s \leq \varepsilon - 1\,;
\end{cases}$$

(6.6.6a)

with

$$\widetilde{\Phi}'_\varepsilon(s) = \Phi'_\varepsilon(s) + \tfrac{\lambda}{2}\,\ln(\tfrac{1-⨍\, w^0}{1+⨍\, w^0})\,, \tag{6.6.6b}$$

and

$$\widetilde{\Phi}''_\varepsilon(s) = \Phi''_\varepsilon(s)\,. \tag{6.6.6c}$$

It is easily established from (6.6.6a)-(6.6.6c), (6.2.2), (6.2.5) and the assumption on w^0 that for any $\varepsilon \in (0, 1-l)$

$$\begin{aligned}\widetilde{\Phi}_\varepsilon(s) &= \widetilde{\Phi}_\varepsilon(⨍\, w^0) + (s - ⨍\, w^0)\,\widetilde{\Phi}'_\varepsilon(⨍\, w^0) + \tfrac{1}{2}\,(s - ⨍\, w^0)^2\,\Phi''_\varepsilon(\xi)\\ &\geq \tfrac{\lambda}{2}\,(s - ⨍\, w^0)^2\,. \end{aligned}\tag{6.6.7}$$

Now, choosing $\eta \equiv \widetilde{\Phi}'_\varepsilon(w) \in L^2(0,T;H^1(\Omega))$ in (6.4.11a) and $\eta \equiv z - ⨍\, z^0 \in L^2(0,T;H^1(\Omega))$ in (6.4.11b) yields after summing the resulting equations and noting (6.1.2) that

$$\begin{aligned}\Big(\widetilde{\Phi}_\varepsilon(w(T)),1\Big) + \tfrac{1}{2}\,\|z(T) - ⨍\, z^0\|_0^2 + \int_0^T \big[\,\rho\,(\Phi''_\varepsilon(w)\nabla w, \nabla w) + |z|_1^2\,\big]\;\mathrm{d}t\\ = \Big(\widetilde{\Phi}_\varepsilon(w^0),1\Big) + \tfrac{1}{2}\,\|z^0 - ⨍\, z^0\|_0^2 + \lambda\int_0^T (\,[\mathcal{V}(w)\,\Phi''_\varepsilon(w) - 1]\,\nabla w, \nabla z)\;\mathrm{d}t\\ + \int_0^T \big(\,\mu\, w - z\,,\, z - ⨍\, z^0\big)\;\mathrm{d}t\,. \end{aligned}\tag{6.6.8}$$

We use (6.2.5) and Lemma 6.6.1 and note (6.6.5) to obtain that

$$\begin{aligned}\rho\,(\Phi''_\varepsilon(w)\nabla w, \nabla w) + |z|_1^2 &\geq \rho\,\lambda\,|w|_1^2 + |z|_1^2\\ &\geq \tfrac{\pi^2\rho\lambda}{L^2}\,\|w - ⨍\, w^0\|_0^2 + \tfrac{\pi^2}{L^2}\,\|z - ⨍\, z^0\|_0^2\,. \end{aligned}\tag{6.6.9}$$

It also follows from (6.6.5), the Cauchy-Schwarz inequality and the Young's inequality for any $\delta > 0$ that

$$\begin{aligned}\big(\,\mu\, w - z\,,\, z - ⨍\, z^0\big) &= \mu\,\big(w - ⨍\, w^0\,,\, z - ⨍\, z^0\big) - \|z - ⨍\, z^0\|_0^2\\ &\leq \mu\,\|w - ⨍\, w^0\|_0\,\|z - ⨍\, z^0\|_0 - \|z - ⨍\, z^0\|_0^2\\ &\leq \tfrac{\delta}{2}\,\|w - ⨍\, w^0\|_0^2 + (\tfrac{\mu^2}{2\,\delta} - 1)\,\|z - ⨍\, z^0\|_0^2\,. \end{aligned}\tag{6.6.10}$$

Combining (6.6.8)-(6.6.10) and noting (6.6.7) leads to

$$\begin{aligned}
\tfrac{\lambda}{2}\,\|w(T) - \textstyle\fint w^0\|_0^2 + \tfrac{1}{2}\,\|z(T) - \textstyle\fint z^0\|_0^2 &\leq \left(\widetilde{\Phi}_\varepsilon(w^0), 1\right) + \tfrac{1}{2}\,\|z^0 - \textstyle\fint z^0\|_0^2 \\
&\quad + \left(\tfrac{\delta}{2} - \tfrac{\pi^2\rho\,\lambda}{L^2}\right)\int_0^T \|w - \textstyle\fint w^0\|_0^2 \,\mathrm{d}t \; + \left(\tfrac{\mu^2}{2\,\delta} - \tfrac{L^2+\pi^2}{L^2}\right)\int_0^T \|z - \textstyle\fint z^0\|_0^2 \,\mathrm{d}t \\
&\quad + \lambda \int_0^T \|\mathcal{V}(w)\,\Phi_\varepsilon''(w) - 1\|_{0,\infty}\,|w|_1\,|z|_1 \,\mathrm{d}t\,. \qquad (6.6.11)
\end{aligned}$$

By letting $\varepsilon \to 0$, we infer from (6.6.11) that

$$\begin{aligned}
&\lambda\,\|w(T) - \textstyle\fint w^0\|_0^2 + \|z(T) - \textstyle\fint z^0\|_0^2 \\
&\qquad \leq C_0 - \lambda\,\delta_1 \int_0^T \|w - \textstyle\fint w^0\|_0^2 \,\mathrm{d}t \, - \delta_2 \int_0^T \|z - \textstyle\fint z^0\|_0^2 \,\mathrm{d}t\,, \qquad (6.6.12)
\end{aligned}$$

where

$$C_0 = \lambda\,\left((1 + w^0)\ln(\tfrac{1+w^0}{1+\fint w^0}) + (1 - w^0)\ln(\tfrac{1-w^0}{1-\fint w^0})\,, 1\right) + \|z^0 - \textstyle\fint z^0\|_0^2\,.$$

and

$$\delta_1 = \frac{2\,\pi^2\rho}{L^2} - \frac{\delta}{\lambda} \quad , \quad \delta_2 = \frac{2\,(L^2 + \pi^2)}{L^2} - \frac{\mu^2}{\delta}\,.$$

On choosing

$$\frac{\mu^2 L^2}{2\,(L^2 + \pi^2)} < \delta < \frac{2\,\pi^2\rho\,\lambda}{L^2}\,,$$

we clearly have $\delta_1\,,\,\delta_2 > 0$. We note that the above choice of δ is possible due to the condition (6.6.3). Now, we have from (6.6.12) that

$$\begin{aligned}
\|w(T) - \textstyle\fint w^0\|_0^2 \quad &\leq \tfrac{C_0}{\lambda} - \delta_1 \int_0^T \|w - \textstyle\fint w^0\|_0^2 \,\mathrm{d}t\,, \\
\|z(T) - \textstyle\fint z^0\|_0^2 \quad &\leq C_0 - \delta_2 \int_0^T \|z - \textstyle\fint z^0\|_0^2 \,\mathrm{d}t\,.
\end{aligned} \qquad (6.6.13)$$

Thus, the desired result (6.6.4) follows easily from (6.6.13). □

Remark 6.6.1 If one were to assume that the integral form of the Grönwall lemma (2.1.5) was applicable on the inequalities in (6.6.13), one would conclude that

$$\begin{aligned}
\|w(T) - \textstyle\fint w^0\|_0^2 \quad &\leq \tfrac{C_0}{\lambda}\,e^{-\delta_1\,T}, \\
\|z(T) - \textstyle\fint z^0\|_0^2 \quad &\leq C_0\,e^{-\delta_2\,T},
\end{aligned} \qquad (6.6.14)$$

which would give exponential decay of the solutions to steady-state constants as $T \to \infty$. In fact, this is not accurate as the integral form of the Grönwall lemma does not consider negative coefficient of the integral in the right hand side (see the discussion in Emmrich [31]). Based on such a wrong use of the Grönwall lemma, similar results to (6.6.14) has been established incorrectly by Galiano *et al.* in [34]. In contrast to [34], in the following example we show that (6.6.13) does not necessarily imply (6.6.14). In other words, we shall show that the non-negativity of the function $v(s)$ in (2.1.5) is crucial.

Example 6.6.1 Consider the following $L^\infty(0,T)$ function

$$u(t) = \begin{cases} e^{-1.1\,t} & \text{if } \ 0 \le t \le r, \\ e^{-1.1\,r} & \text{if } \ r \le t \le \frac{13\,r}{11}\,, \\ e^{-t} & \text{if } \ t > \frac{13\,r}{11}\,, \end{cases}$$

where $r \ge 1$ is fixed. We calculate for *a.e.* $t \in (0,T)$

$$f(t) := u(t) + \int_0^t u(s)\ \mathrm{d}s\,.$$

For $0 \le t \le r$, we note that $f'(t) = u'(t) + u(t) = -0.1\,e^{-1.1\,t} < 0$, so f in monotone decreasing. This implies $u(t) \le u(0) = 1$. In fact,

$$\begin{aligned} u(t) + \int_0^t u(s)\ \mathrm{d}s \ &= e^{-1.1\,t} + \int_0^t e^{-1.1\,s}\ \mathrm{d}s \\ &= e^{-1.1\,t} + \tfrac{10}{11}\,(1 - e^{-1.1\,t}) \\ &= \tfrac{1}{11}\,(10 + e^{-1.1\,t}) \le 1\,. \end{aligned} \tag{6.6.15}$$

For $r \le t \le \frac{13\,r}{11}$, we have that $f'(t) = e^{-1.1\,r} > 0$ and f increases linearly. In fact,

$$\begin{aligned} u(t) + \int_0^t u(s)\ \mathrm{d}s \ &= e^{-1.1\,r} + \int_0^r e^{-1.1\,s}\ \mathrm{d}s\ + \int_r^t e^{-1.1\,r}\ \mathrm{d}s \\ &= e^{-1.1\,r} + \tfrac{10}{11}\,(1 - e^{-1.1\,r}) + e^{-1.1\,r}(t-r) \\ &\le e^{-1.1\,r} + \tfrac{10}{11}\,(1 - e^{-1.1\,r}) + \tfrac{2\,r}{11}\,e^{-1.1\,r} \\ &= \tfrac{1+2\,r}{11}\,e^{-1.1\,r} + \tfrac{10}{11} \le 1\,. \end{aligned} \tag{6.6.16}$$

For $t > \frac{13\,r}{11}$, $f'(t) = 0$ and

$$\begin{aligned} u(t) + \int_0^t u(s)\;\mathrm{d}s \; &= e^{-t} + \int_0^r e^{-1.1\,s}\;\mathrm{d}s \; + \int_r^{\frac{13\,r}{11}} e^{-1.1\,r}\;\mathrm{d}s \; + \int_{\frac{13\,r}{11}}^t e^{-s}\;\mathrm{d}s \\ &= \tfrac{10}{11}\,(\,1 - e^{-1.1\,r}) + \tfrac{2\,r}{11}\,e^{-1.1\,r} + e^{\frac{-13\,r}{11}} \\ &< \tfrac{10}{11}\,(\,1 - e^{-1.1\,r}) + \tfrac{2\,r}{11}\,e^{-1.1\,r} + e^{-1.1\,r} \\ &= \tfrac{1+2\,r}{11}\,e^{-1.1\,r} + \tfrac{10}{11} \le 1\,. \end{aligned} \tag{6.6.17}$$

Thus we conclude from (6.6.15), (6.6.16) and (6.6.17) that for *a.e.* $t \in (0, T)$

$$u(t) + \int_0^t u(s)\;\mathrm{d}s \;\le u(0)$$

and yet $u(t) \not\leq u(0)\,e^{-t}$ in the range $t \in (\frac{11\,r}{10}, \frac{13\,r}{11})$.

Although the above example shows that (6.6.14) is not generally valid, the convergence of $u(t)$ to zero is satisfied since we have for $t > \frac{13\,r}{11}$ that

$$u(t) = e^{-t} \longrightarrow 0 \quad \text{as } t \to \infty\,.$$

We also note that choosing r larger delays the exponential decay of $u(t)$.

Further, the following example indicates that inequalities such as (6.6.13) do not necessarily lead to the convergence of w and z to their mean integrals.

Example 6.6.2 Define u to be the $L^\infty(0,T)$ function given by

$$u(t) = \begin{cases} C & \text{for } \; t \in [2^n\,,\, 2^n + 2^{-n}]\,,\; n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We have for any $t > 0$ that

$$u(t) + \int_0^t u(s)\;\mathrm{d}s \;\le C + \int_0^\infty u(s)\;\mathrm{d}s = C + C\sum_{n=1}^\infty 2^{-n} = 2C\,,$$

which is identical to (6.6.13). But the function $u(t)$ does not converge to 0 as $t \to \infty$.

We close the discussion on Theorem 6.6.2 by giving the following remark:

Remark 6.6.2 We note that the stability condition (6.6.3) requires the domain L to be sufficiently small. On the other hand, the linear stability study of the linearized problem of (Q), presented by Aranson *et al.* [4], reveals that for $\mu > \rho$ long-wave perturbations are unstable. In fact, it has been shown in [4] that perturbations of the form $e^{\sigma t + i k x}$, where $\sigma \in \mathbb{R}$ is the growth rate and $k \in \mathbb{R}$ is the wavenumber of the perturbation, are unstable only if $\mu > \rho$ and

$$k^2 < \frac{\mu - \rho}{\lambda + \rho}\,.$$

Thus, as observed in [34], for $k = \frac{2\pi}{L}$ the condition becomes

$$L^2 > 4\,\pi^2\,\frac{\lambda + \rho}{\mu - \rho}\,. \tag{6.6.18}$$

It can be easily checked that the condition (6.6.3) does not allow (6.6.18) to be satisfied. This indicates that segregation is more likely if the length L of the domain is large enough. In fact, it is pointed out in [3] that the segregation phenomena is expected to occur in a long rotating drum.

For further discussion, we consider the linearized equations of (1.2.1a)-(1.2.1b) about the origin with appropriate boundary conditions and initial data:

$$\frac{\partial w}{\partial t} = \nabla\cdot(\,\rho\,\nabla w - \nabla z\,)\,, \tag{6.6.19a}$$

$$\frac{\partial z}{\partial t} = \nabla\cdot(\,\nabla z + \lambda\,\nabla w\,) + \mu\,w - z\,. \tag{6.6.19b}$$

Multiplying the equations (6.6.19a,b) by $\lambda\,(w - ⨍\, w^0)$ and $(z - ⨍\, z^0)$ respectively, integrating by parts over Ω and summing the resulting equations yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{\lambda}{2}\,||w - ⨍\, w^0||_0^2 + \tfrac{1}{2}||z - ⨍\, z^0||_0^2\right) + \rho\,\lambda\,|w|_1^2 + |z|_1^2 = \big(\mu\,w - z, z - ⨍\, z^0\big)\,. \tag{6.6.20}$$

Assuming that $\mu ⨍\, w^0 = ⨍\, z^0$ and noting the second inequality in (6.6.9) and (6.6.10) gives, under the condition (6.6.3), that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda\,||w - ⨍\, w^0||_0^2 + ||z - ⨍\, z^0||_0^2\right) &\le -\lambda\,\delta_1\,||w - ⨍\, w^0||_0^2 - \delta_2\,||z - ⨍\, z^0||_0^2 \\ &\le -\delta_*\left(\lambda\,||w - ⨍\, w^0||_0^2 + ||z - ⨍\, z^0||_0^2\right),\end{aligned} \tag{6.6.21}$$

where δ_1 and δ_2 are the same parameters defined in Theorem 6.6.2 and $\delta_* = \min\{\delta_1\,,\,\delta_2\} > 0$. Multiplying (6.6.21) by $e^{\delta_* t}$ and integrating over $(0,T)$ leads to

$$\lambda\,||w(T) - ⨍\, w^0||_0^2 + ||z(T) - ⨍\, z^0||_0^2 \le C_*\, e^{-\delta_* T}, \tag{6.6.22}$$

where $C_* = \lambda\,||w^0 - ⨍\, w^0||_0^2 + ||z^0 - ⨍\, z^0||_0^2$. Thus, under the assumption (6.6.3), we conclude that the linearized solution $\{w, z\}$ decays exponentially fast towards the stationary solution $\{⨍\, w^0, ⨍\, z^0\}$ in L^2-norm.

From the above linear stability discussion, one expects if the condition (6.6.3) holds we will have non-growth solutions of (Q). Actually, in agreement with the finite difference experiments in [34], our numerical simulations in the next chapter show that under the conditions of Theorem 6.6.2 the finite element approximations of w and z converge to the mean integrals of the initial data w^0 and z^0 respectively. However, this does not contradict Example 6.6.2 as there might be alternative mathematical techniques that can be used to treat the proof of Theorem 6.6.2 differently in order to conclude an explicit convergence of the solutions w and z to steady-state constants.

The following remark is related to the boundary conditions of problem (Q):

Remark 6.6.3 Our analysis of problem (Q) also works if we consider, instead of (1.2.1c), the periodic boundary conditions in (6.1.4). In this case, the analysis will be exactly the same except we only need to replace the finite element space S^h by the space S^h_p which is generated by the basis functions $\{\varphi_{p,j}\}_0^{J-1}$ defined by:

$$\varphi_{p,0} := \varphi_0 + \varphi_J \quad \text{and} \quad \varphi_{p,j} := \varphi_j \quad \text{for} \quad j = 1, \cdots,\ J-1\,,$$

where $\{\varphi_j\}_0^J$ is the canonical basis associated with S^h. The lumped mass matrix and the stiffness matrix corresponding to the spaces S^h and S^h_p can be found in Appendix A.2.

Chapter 7

The axial segregation model: Numerical experiments

In this chapter we shall perform some numerical experiments for problem (Q). We first state a practical algorithm for solving the approximate problem ($\mathrm{Q}_\varepsilon^{h,\Delta t}$). Then we establish and discuss some numerical solutions. As in Chapter 5, the programs are written in Fortran, see Appendix B.2, and the graphs are generated in Matlab.

7.1 Numerical results

To solve the resulting system of nonlinear algebraic equations, for $\{W_\varepsilon^n, Z_\varepsilon^n\}$, arising at each time level from the approximation (6.3.6a)-(6.3.6b), we use the following iterative approach:
Given $W_\varepsilon^{n,0} \in S^h$, for $k \geq 1$ find $\{W_\varepsilon^{n,k}, Z_\varepsilon^{n,k}\} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$\left(\tfrac{W_\varepsilon^{n,k}-W_\varepsilon^{n-1}}{\Delta t_n}, \chi\right)^h + \rho\left(\nabla W_\varepsilon^{n,k}, \nabla \chi\right) - \lambda\left(\Pi_\varepsilon(W_\varepsilon^{n,k-1})\,\nabla Z_\varepsilon^{n,k}, \nabla \chi\right) = 0\,, \qquad (7.1.1\text{a})$$

$$\left(\tfrac{Z_\varepsilon^{n,k}-Z_\varepsilon^{n-1}}{\Delta t_n}, \chi\right)^h + \left(Z_\varepsilon^{n,k}, \chi\right)^h + \left(\nabla Z_\varepsilon^{n,k}, \nabla \chi\right) + \lambda\left(\nabla W_\varepsilon^{n,k}, \nabla \chi\right) = \mu\left(\theta\, W_\varepsilon^{n,k} + (1-\theta)\, W_\varepsilon^{n-1}, \chi\right)^h. \qquad (7.1.1\text{b})$$

For the iterative algorithm (7.1.1a)-(7.1.1b), we start with $W_\varepsilon^0 \equiv \pi^h w^0$ and $Z_\varepsilon^0 \equiv \pi^h z^0$, and we set, for $n \geq 1$, $W_\varepsilon^{n,0} \equiv W_\varepsilon^{n-1}$ and $Z_\varepsilon^{n,0} \equiv Z_\varepsilon^{n-1}$. We also choose

$TOL = 1 \times 10^{-7}$ and adopt the stopping criteria

$$\left|W_\varepsilon^{n,k} - W_\varepsilon^{n,k-1}\right|_{0,\infty} < TOL \quad \text{and} \quad \left|Z_\varepsilon^{n,k} - Z_\varepsilon^{n,k-1}\right|_{0,\infty} < TOL. \tag{7.1.2}$$

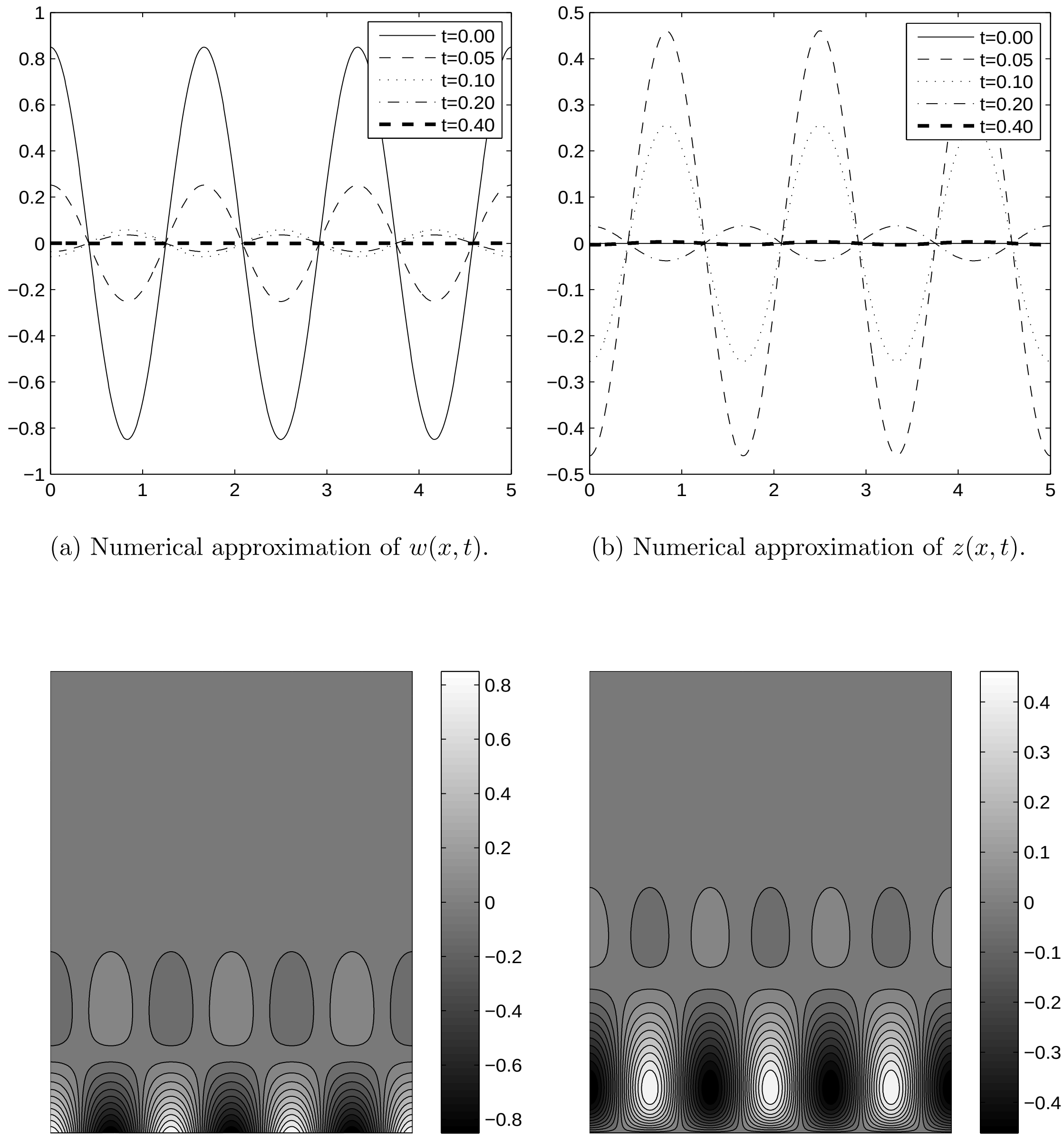
Hence, for k satisfying (7.1.2), we set $W_\varepsilon^n \equiv W_\varepsilon^{n,k}$ and $Z_\varepsilon^n \equiv Z_\varepsilon^{n,k}$.

Algebraically, the existence of the solutions of the system (7.1.1a)-(7.1.1b) can be easily investigated on noting the fact that for a square linear system existence is equivalent to uniqueness. Practically, to solve the above scheme at each iteration, we use the basis functions of the space S^h to construct a linear system that can be solved efficiently through linear programming. Later on, in this chapter, we perform some experiments for discrete periodic boundary conditions; that is when we consider the space S_p^h, instead of S^h, in the above iterative algorithm. Although we have no convergence proof for the iteration (7.1.1a)-(7.1.1b), good convergence properties have been observed in practice.

Unless otherwise stated, in all simulations we consider a uniform partitioning of $\Omega = (0, L)$, with mesh points $x_j = j\,h,\ j = 0, \cdots, J$, and uniform time steps $t_n = n\,\Delta t,\ n = 1, \cdots, N$. We take $J = 512$ (i.e. $h = \frac{L}{512}$), $N = 1000$ (i.e. $\Delta t = \frac{T}{1000}$) and $\varepsilon = 1 \times 10^{-9}$. We also consider, as in [4] and [34], the initial conditions $w^0(x) = \zeta\,\cos(k\,x)$ and $z^0(x) = 0$ for real numbers ζ and k.

For the first experiment we took the parameters $\lambda = 2,\ \rho = 1,\ \mu = 2,\ \theta = 1,\ L = 5$ and $T = 0.5$ with initial preseparated state determined by $\zeta = 0.85$ and $k = \frac{6\,\pi}{L}$. The numerical solutions are plotted in Figure 7.1(a)-(b) at several times. Since the parameters satisfy the condition (6.6.3), one expects non-growth solutions. In agreement with what we expect, the solutions in Figure 7.1(a)-(b) decay to zero as t increases. The overall description of the numerical solutions, W_ε^n and Z_ε^n, $n \geq 1$, can be seen in Figure 7.1(c)-(d). Obviously, the granular materials do not segregate.

To see segregation behaviour, we kept all parameters the same as the previous experiment, except $L = 35$ and $T = 50$. The solutions are plotted in Figure 7.2(a)-(b) at several times, and fully described in Figure 7.2(c)-(d). In this experiment, the length L of the drum was large enough to allow the grains to segregate leading to a stable array of concentration bands; see Figure 7.2(c).



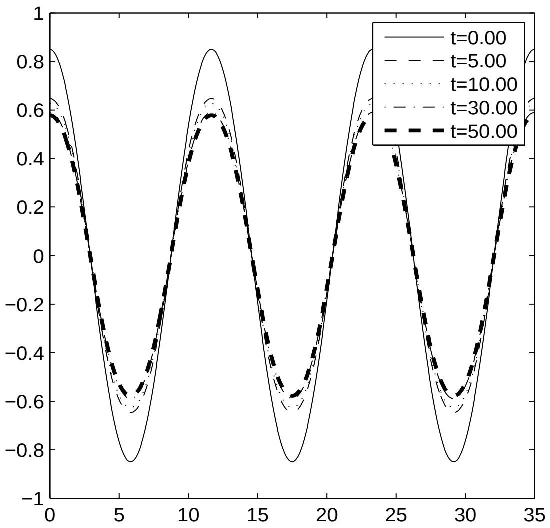
(a) Numerical approximation of $w(x,t)$.

(b) Numerical approximation of $z(x,t)$.

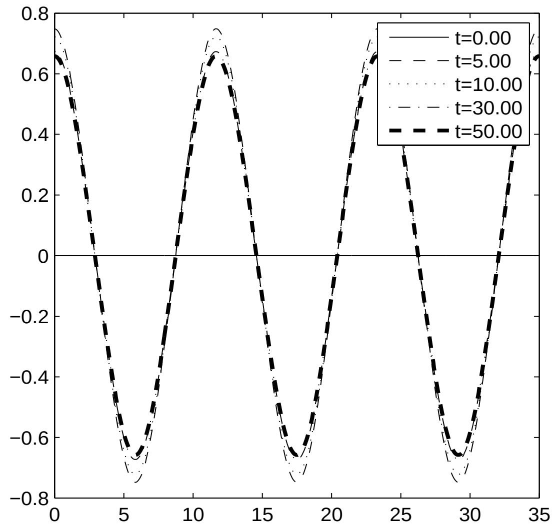
(c) Numerical approximation of $w(x,t)$.

(d) Numerical approximation of $z(x,t)$.

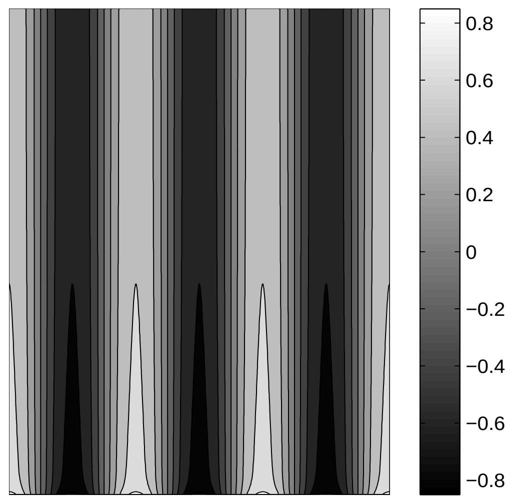
Figure 7.1: Numerical solution of problem (Q). The parameters are: $\lambda = 2$, $\rho = 1$, $\mu = 2$, $\theta = 1$, $L = 5$, $T = 0.5$, $N = 1000$, $\zeta = 0.85$ and $k = \frac{6\pi}{L}$. In (a) and (b) the numerical solutions are given at several times. In (c) and (d) the numerical solutions are presented in the (x,t)-plane for $0 \leq x \leq L$ and $0 \leq t \leq T$.



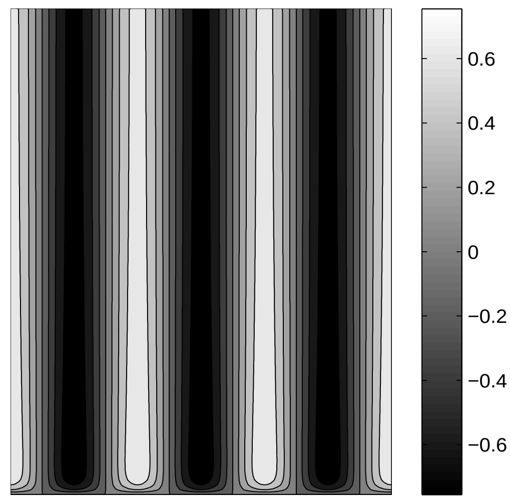
(a) Numerical approximation of $w(x,t)$.



(b) Numerical approximation of $z(x,t)$



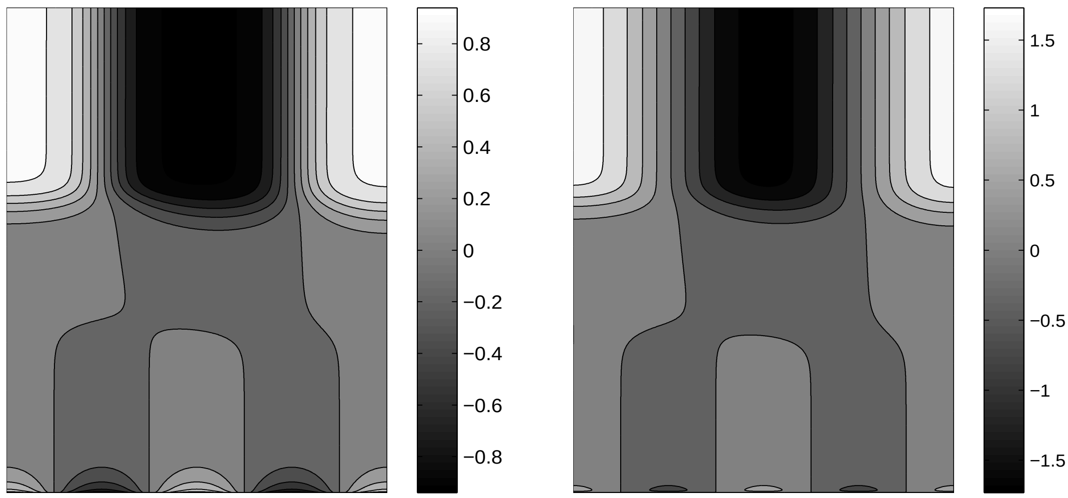
(c) Numerical approximation of $w(x,t)$.



(d) Numerical approximation of $z(x,t)$.

Figure 7.2: Numerical solution of problem (Q). The parameters are: $\lambda = 2,\ \rho = 1,\ \mu = 2,\ \theta = 1,\ L = 35,\ T = 50,\ N = 1000,\ \zeta = 0.85$ and $k = \frac{6\pi}{L}$. In (a) and (b) the numerical solutions are given at several times. In (c) and (d) the numerical solutions are presented in the (x,t)-plane.

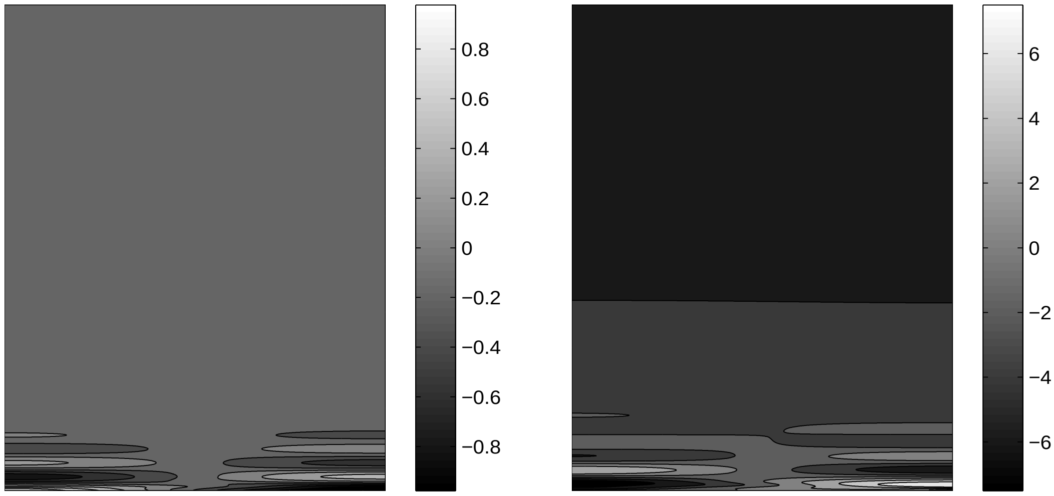
To see another interesting behaviour, we repeat the above experiment with the same parameters, except $k = \frac{4\pi}{L}$, $L = 20$ and $T = 500$. The solutions are plotted in Figure 7.3(a)-(b). It can be seen from Figure 7.3(a) that the initial perturbation decays and after long time the separation starts to occur again obtaining well-segregated bands.



(a) Numerical approximation of $w(x,t)$. (b) Numerical approximation of $z(x,t)$.

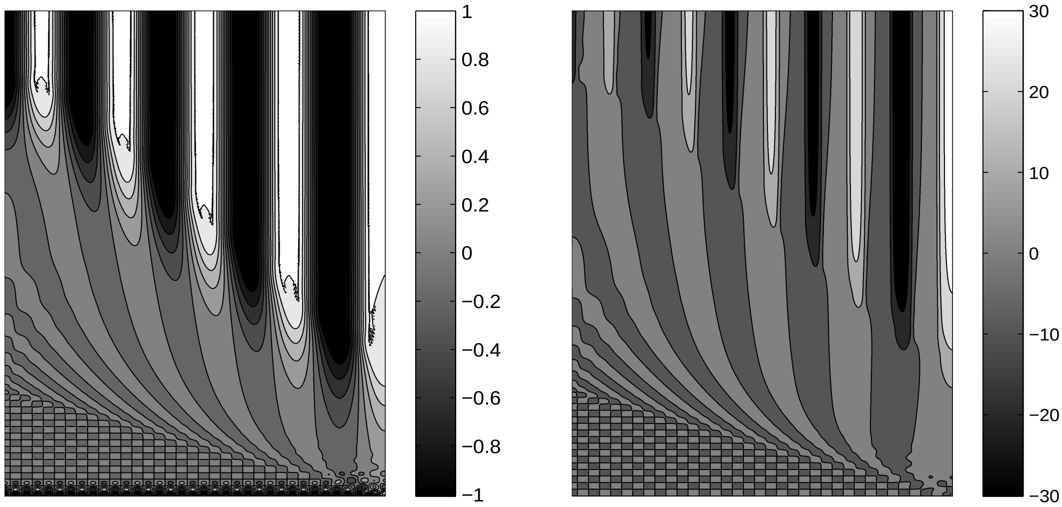
Figure 7.3: Numerical solution of problem (Q) in the (x,t)-plane. The parameters are: $\lambda = 2,\ \rho = 1,\ \mu = 2,\ \theta = 1,\ L = 20$, $T = 500$, $N = 1000$, $\zeta = 0.85$ and $k = \frac{4\pi}{L}$.

We also solved the iteration (7.1.1a)-(7.1.1b) for the following parameters: $\lambda = 100,\ \rho = 0.5,\ \mu = 40,\ \theta = 1,\ \zeta = 0.95$ and $k = 1.8$. In Figure 7.4 (a) and (b) respectively, the solutions W_ε^n and Z_ε^n are plotted for $L = 2$ and $0 \le t \le 5$. The resulting convergence behaviour of the solutions, in Figure 7.4(a)-(b), is expected from the discussion on Theorem 6.6.2. In Figure 7.5(a)-(b), we did the same experiment but for $L = 60$ and $0 \le t \le 8$. In this case, the initial perturbation produces decaying standing waves which are replaced later by ten segregated bands, see Figure 7.5(a).



(a) Numerical approximation of $w(x,t)$. (b) Numerical approximation of $z(x,t)$.

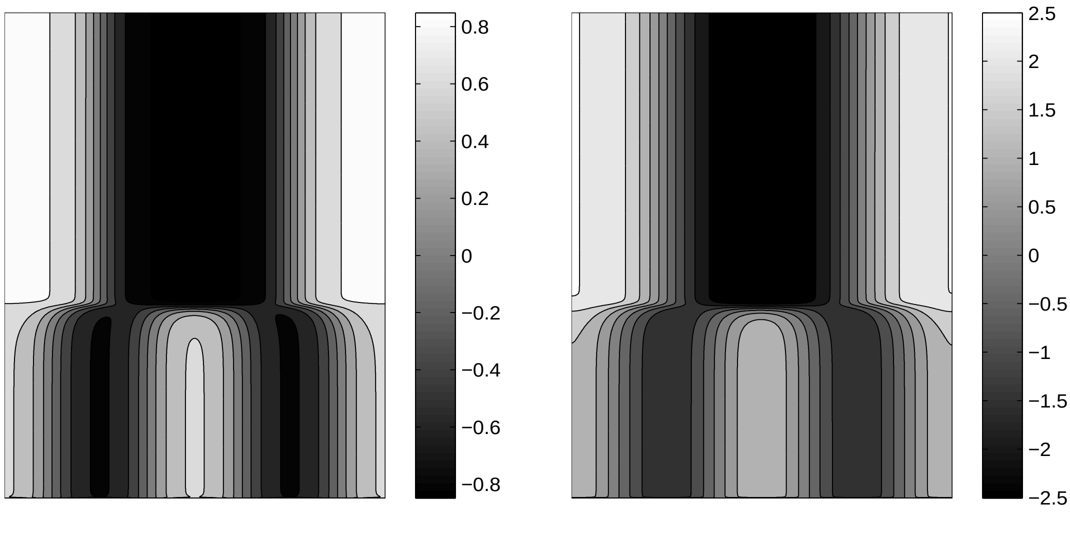
Figure 7.4: Numerical solution of problem (Q) in the (x,t)-plane. The parameters are: $\lambda = 100,\ \rho = 0.5,\ \mu = 40,\ \theta = 1,\ L = 2,\ T = 5,\ N = 1000,\ \zeta = 0.95$ and $k = 1.8$.



(a) Numerical approximation of $w(x,t)$. (b) Numerical approximation of $z(x,t)$.

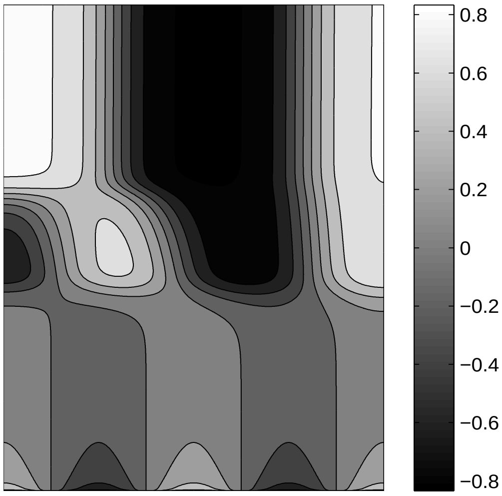
Figure 7.5: Numerical solution of problem (Q) in the (x,t)-plane. The parameters are: $\lambda = 100,\ \rho = 0.5,\ \mu = 40,\ \theta = 1,\ L = 60,\ T = 8,\ N = 1000,\ \zeta = 0.95$ and $k = 1.8$.

In Figure 7.6(a)-(b) we repeated an experiment performed in [34]. We took $\lambda = 2$, $\rho = 2$, $\mu = 3$, $\theta = 1$, $\zeta = 0.8$ and $k = \frac{4\pi}{L}$ with $L = 30$, $T = 4000$ and $N = 10000$. The same experiment, with $L = 25$ and $T = 1000$, is performed in Figure 7.7(a)-(b). Obviously, the solutions follow different behaviour than those in Figure 7.6(a)-(b). In fact, both the length of the drum, L, and the wavenumber of the initial state, k, have an influence on the dynamics of the materials. For instance, we note that increasing the length of the drum allows more bands to emerge; see Figure 7.8(a)-(b). We also note that in the early stages of evolution the solutions in Figure 7.9(a)-(b) have different families of standing waves than Figure 7.7(a)-(b); but later the solutions become identical. In agreement with the linear stability analysis in [4], we found experimentally that the condition $\mu > \rho$ needs to be satisfied in order to have size segregation. In this respect, we ran many experiments for different choices of ρ and μ such that $\mu \leq \rho$ and never observed segregation behaviour.

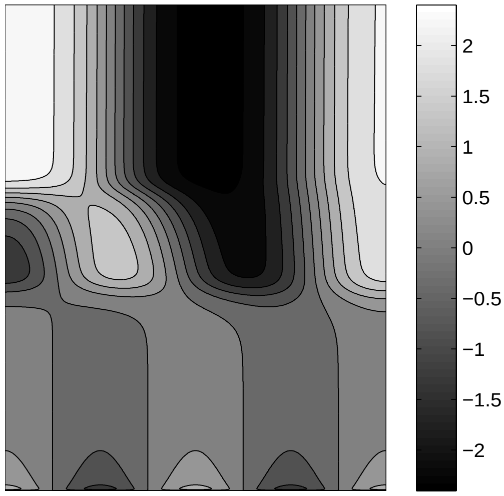


(a) Numerical approximation of $w(x, t)$. (b) Numerical approximation of $z(x, t)$.

Figure 7.6: Numerical solution of problem (Q) in the (x, t)-plane. The parameters are: $\lambda = 2$, $\rho = 2$, $\mu = 3$, $\theta = 1$, $L = 30$, $T = 4000$, $N = 10000$, $\zeta = 0.8$ and $k = \frac{4\pi}{L}$.

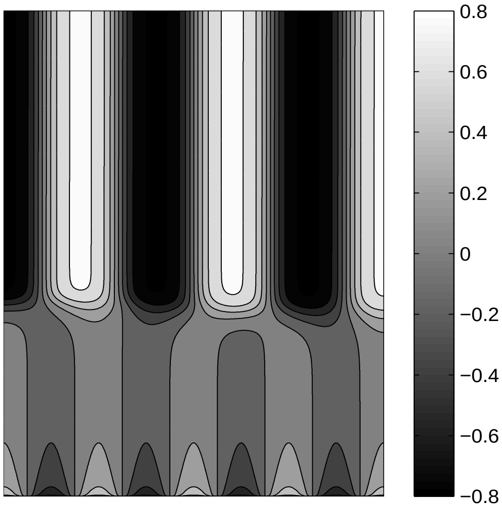


(a) Numerical approximation of $w(x,t)$.

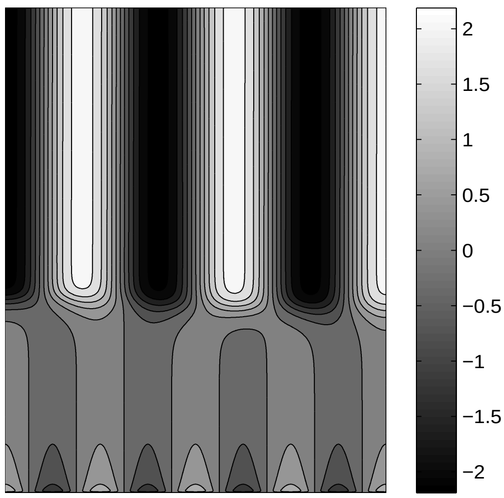


(b) Numerical approximation of $z(x,t)$.

Figure 7.7: Numerical solution of problem (Q) in the (x,t)-plane. The parameters are: $\lambda = 2,\ \rho = 2,\ \mu = 3,\ \theta = 1,\ L = 25,\ T = 1000,\ N = 10000,\ \zeta = 0.8$ and $k = \frac{4\pi}{L}$.

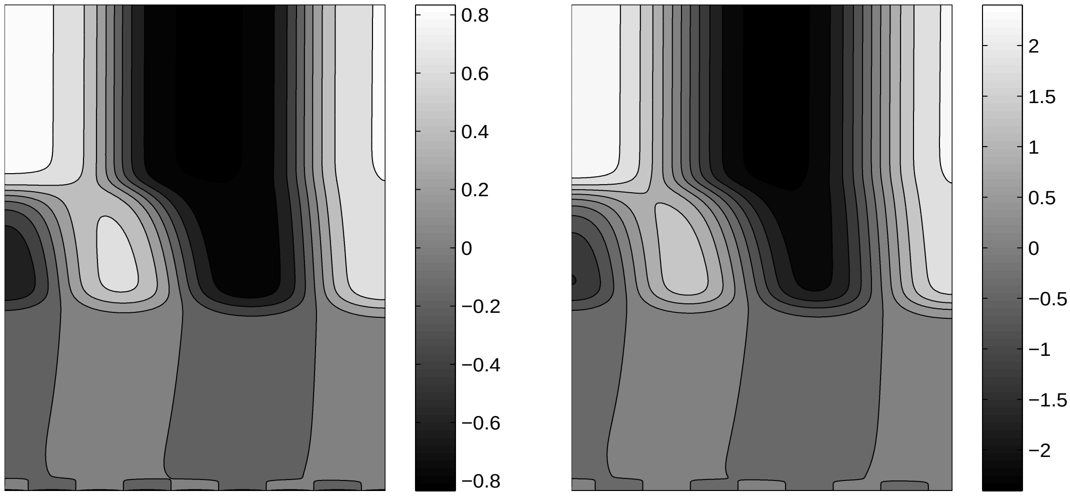


(a) Numerical approximation of $w(x,t)$.



(b) Numerical approximation of $z(x,t)$.

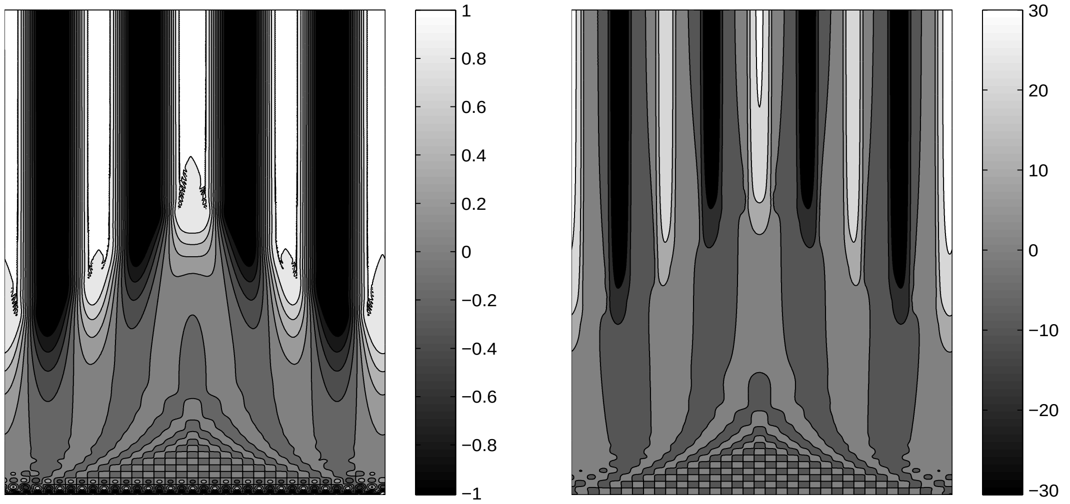
Figure 7.8: Numerical solution of problem (Q) in the (x,t)-plane. The parameters are: $\lambda = 2,\ \rho = 2,\ \mu = 3,\ \theta = 1,\ L = 50,\ T = 1000,\ N = 10000,\ \zeta = 0.8$ and $k = \frac{8\pi}{L}$.



(a) Numerical approximation of $w(x,t)$. (b) Numerical approximation of $z(x,t)$.

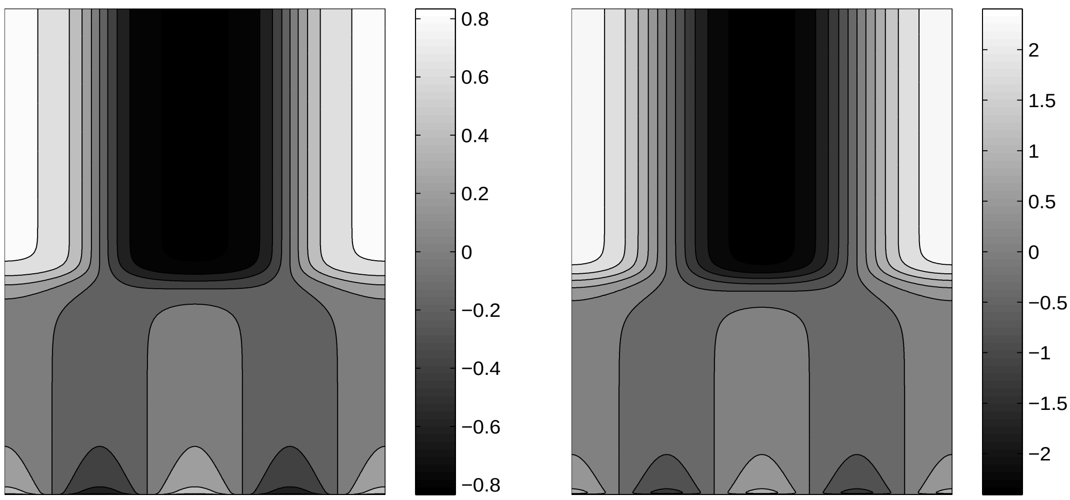
Figure 7.9: Numerical solution of problem (Q) in the (x,t)-plane. The parameters are: $\lambda = 2,\ \rho = 2,\ \mu = 3,\ \theta = 1,\ L = 25,\ T = 1000,\ N = 10000,\ \zeta = 0.8$ and $k = \frac{8\pi}{L}$.

As mentioned in Remark 6.6.3, the analysis presented in Chapter 6 also works for periodic boundary conditions. In order to compare the influence of the boundary conditions on the dynamics of the granular materials, we have repeated some of the above experiments with the consideration of discrete periodic boundary conditions. Figure 7.10 shows the numerical solutions corresponding to Figure 7.5. The effect of the periodic boundary conditions is obvious. Repeating the experiment in Figure 7.10 with double size domain, i.e. $L = 120$, we obtain periodic solutions which are identical to the solutions in Figure 7.5 for $60 \le x \le 120$. The influence of the periodic boundary conditions can be also seen in Figure 7.11 – Figure 7.13 where the solutions corresponding to Figure 7.7 – Figure 7.9 are plotted respectively. As indicated in previous experiment, the solutions in Figure 7.12 for $25 \le x \le 50$ behave similarly to the solutions in Figure 7.7. We also repeated the experiments in Figure 7.2 and Figure 7.6 with periodic boundary conditions, already noting that the Neumann boundary condition solution appears to be periodic. We found the results are graphically identical. We indicate that our numerical results for periodic boundary conditions are in qualitative agreement with the numerical experimental observations in [34].



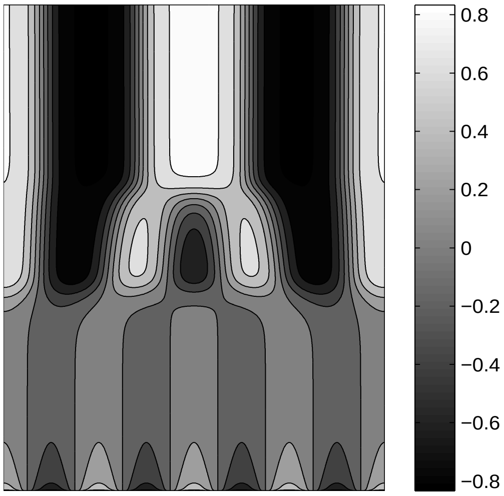
(a) Numerical approximation of $w(x,t)$. (b) Numerical approximation of $z(x,t)$.

Figure 7.10: As in Figure 7.5 but for periodic boundary conditions.

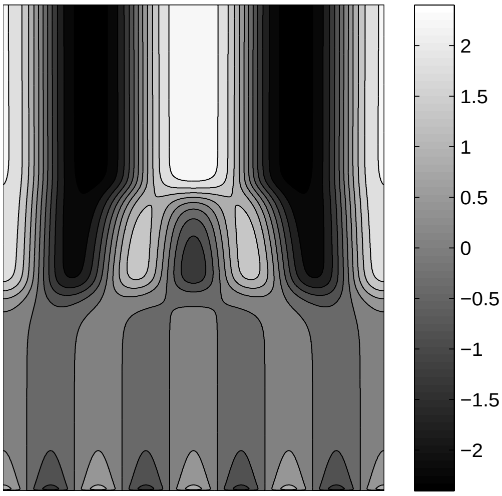


(a) Numerical approximation of $w(x,t)$. (b) Numerical approximation of $z(x,t)$.

Figure 7.11: As in Figure 7.7 but for periodic boundary conditions.

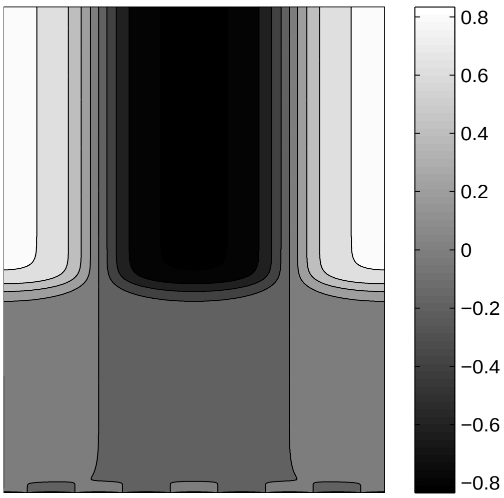


(a) Numerical approximation of $w(x,t)$.

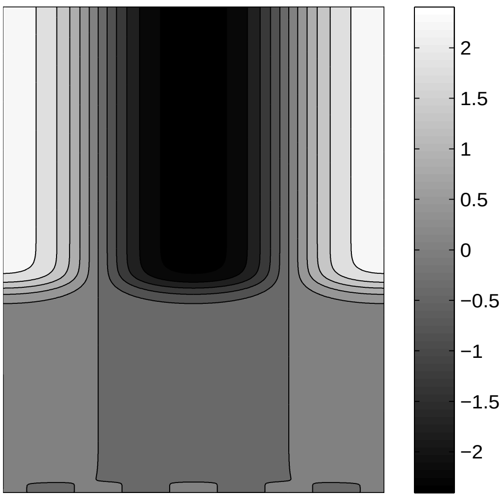


(b) Numerical approximation of $z(x,t)$.

Figure 7.12: As in Figure 7.8 but for periodic boundary conditions.



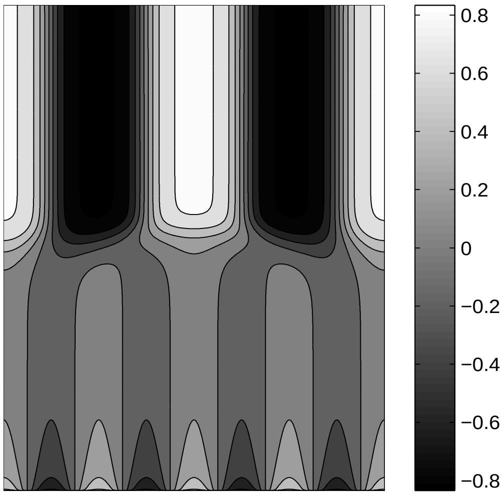
(a) Numerical approximation of $w(x,t)$.



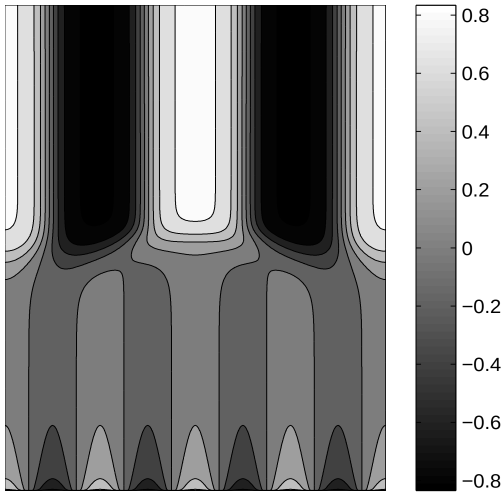
(b) Numerical approximation of $z(x,t)$.

Figure 7.13: As in Figure 7.9 but for periodic boundary conditions.

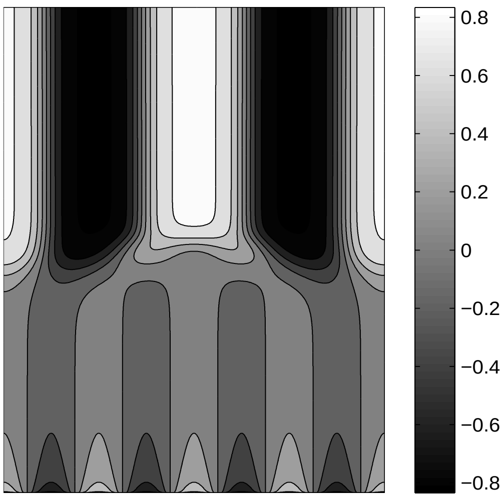
Finally, as an attempt to illustrate how the parameter θ effects the behaviour of the numerical solutions obtained from (7.1.1a)-(7.1.1b), we have repeated the experiment in Figure 7.12 for different values of θ. We remark that the numerical solutions behave differently in the early stages of evolution and in the first stages of the appearance of the stationary bands; see Figure 7.14. As no exact solution to (Q) is known, we can not decide which value of the parameter θ is better.



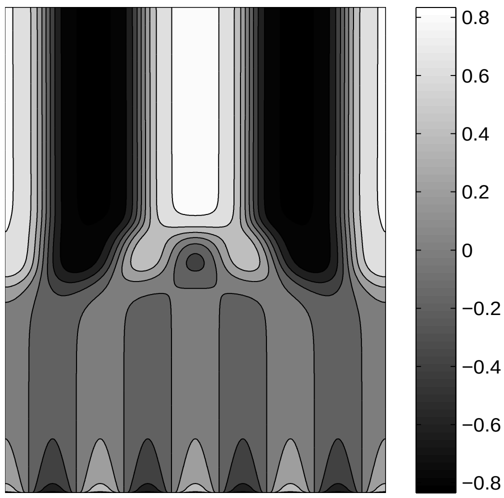
(a) Numerical approximation of $w(x,t)$.



(b) Numerical approximation of $w(x,t)$.



(c) Numerical approximation of $w(x,t)$.



(d) Numerical approximation of $w(x,t)$.

Figure 7.14: The behaviour of w in the (x,t)-plane. The same experiment as in Figure 7.12(a); except (a) $\theta = 0$, (b) $\theta = 0.25$, (c) $\theta = 0.50$ and (d) $\theta = 0.75$.

Chapter 8

Conclusions

We studied two strongly coupled cross diffusion systems using a finite element method. The first system, (P), is a population model which represents the movement of two interacting cell populations in $d \leq 3$ space dimensions. The second system, (Q), is proposed in one space dimension to model the axial segregation of two kinds of granular materials. In the first chapter of the thesis we introduced the models (P) and (Q) and defined the research objectives. Our study of the model (P) was executed in four chapters, Chapter 2, 3, 4 and 5, and the rest of the thesis was devoted to the study of the model (Q).

It was shown using finite element techniques that there exists a global weak solution of the population system (P). A technical replacement was the key to our study of the system where we considered a truncated alternative problem to (P). The singular nature of (P) in $\mathbb{R}^{\leq 0}$ has been treated by employing an appropriate regularization procedure. A well defined entropy inequality of the regularized problem has been derived. A fully discrete finite element approximation to (P) has been introduced. The existence of the fully discrete solutions has been shown for sufficiently small time discretization parameter. An analogous discrete inequality has been obtained and some stability bounds on the approximations have been established. By using sequential compactness arguments, the convergence of the finite element approximate problem has been studied and existence of a non-negative weak solution for (P) was concluded. Further regularity results have been shown for a "fully" truncated alternative problem to (P). In the absence of the reaction terms, some other mathematical results for the model (P) have been discussed. At the end of our study of the population model (P), we successfully performed some numerical experiments in one space dimension that support the established theoretical results.

The mathematical analysis used in proving the existence results for (P) has been briefly adapted to show that there exists a global weak solution of the axial segregation model (Q). Some uniqueness results have been discussed. An error bound between the fully discrete and weak solutions of (Q) has been proved. The long time behaviour of the solutions of (Q) has been investigated. In this respect, a major hole in the work of others has been uncoverd and discussed. Finally, the established theoretical results for the model (Q) have been illustrated by performing some numerical experiments.

Although the axial segregation model (Q) is intrinsically one-dimensional in space, our mathematical analysis of the model can be naturally extended for $d = 2$ and 3. However, as the continuous embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ holds only for $d = 1$, our uniqueness and error bound analysis of (Q) is not valid for multi-dimensional spaces; see (6.4.28) and (6.5.15).

Additional regularity, more than we have been able to prove, was required to complete the uniqueness proof and error bound analysis for problem (Q). Unfortunately, we have been unable to prove the regularity requirement which was essential to establish these results. However, it might be possible and this is left open for future investigation. With regard to the problem (P), a considerable idea for obtaining uniqueness results is by mimicking the uniqueness study presented for the model (Q). In this direction, and due to the structure of the model (P), the analysis will be faced in addition to the regularity requirement by other technical obstacles. This is also left as an open problem for future work.

Numerically, there are remaining issues that can be investigated but because we have limited time we leave them for future study. For example, one could try to perform numerical experiments for the population model in higher space dimensions. We were unable to numerically verify the fully discrete error bound for (Q) because no exact solution is known. However, experimental work that can be done in this direction is comparing the computed solution on a coarse mesh with that on a fine mesh. We also might be able to improve the error estimate by adapting the ideas in Barrett and Blowey [8]. We leave this for future investigation.

In Section 6.6, a note on the use of the Grönwall lemma in [34] was reported. One of the authors of [34] with others have published another paper for studying the long time behaviour of solutions of the viscous quantum hydrodynamic equations, see [37]. The analysis in that paper was mainly based on using the Grönwall lemma, similarly to the wrong application discussed in Section 6.6, to conclude the exponential decay of the solutions. It would be of interest to investigate the consequences of this analytical mistake.

The mathematical work in this thesis can be used to analyse other cross diffusion systems. For example, following similar arguments of replacement to that for (P), one can improve the analysis presented in [21] and [9]. One could also try to adapt the techniques employed in this thesis to study the cross diffusion models in [45] and [39].

Bibliography

[1] R. A. Adams (1975), *Sobolev Spaces*, Pure and Applied Mathematics, Academic Press, New York.

[2] R. A. Adams and J. Fournier (1977), *Cone conditions and properties of Sobolev spaces*, J. Math. Anal. Appl. **61**, pp. 713–734.

[3] I. S. Aranson and L. S. Tsimring (1999), *Dynamics of axial separation in long rotating drums*, Phys. Rev. Lett. **82**, pp. 4643–4646.

[4] I. S. Aranson, L. S. Tsimring and V. M. Vinokur (1999), *Continuum theory of axial segregation in a long rotating drum*, Phys. Rev. E, **60**, pp. 1975–1987.

[5] I. S. Aranson and L. S. Tsimring (2006), *Patterns and collective behavior in granular media: Theoretical concepts*, Rev. Mod. Phys. **78**, pp. 641–692.

[6] C. Baiocchi and A. Capelo (1984), *Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems*, John Wiley and Sons, Chichester.

[7] J. W. Barrett and J. F. Blowey (1999), *Finite element approximation of the Cahn-Hilliard equation with concentration dependent mobility*, Math. Comp. **68**, pp. 487–517.

[8] J. W. Barrett and J. F. Blowey (2001), *An improved error bound for a finite element approximation of a model for phase separation of a multi-component alloy with a concentration dependent mobility matrix*, Numer. Math. **88**, pp. 255–297.

[9] J. W. Barrett and J. F. Blowey (2004), *Finite element approximation of a nonlinear cross-diffusion population model*, Numer. Math. **98**, pp. 195–221.

[10] J. W. Barrett, H. Garcke and R. Nürnberg (2003), *Finite element approximation of surfactant spreading on a thin film*, SIAM J. Numer. Anal. **41**, pp. 1427–1464.

[11] J. W. Barrett and R. Nürnberg (2002), *Finite element approximation of a nonlinear degenerate parabolic system describing bacterial pattern formation*, Interfaces and Free Boundaries **4**, pp. 277–307.

[12] J. W. Barrett and R. Nürnberg (2004), *Convergence of a finite element approximation of surfactant spreading on a thin film in the presence of van der Waals forces*, IMA J. Numer. Anal. **24**, pp. 323–363.

[13] J. W. Barrett, R. Nürnberg and V. Styles (2004), *Finite element approximation of a phase field model for void electromigration*, SIAM J. Numer. Anal. **42**, pp. 738–772.

[14] M. Bebendorf (2003), *A Note on the Poincaré Inequality for Convex Domains*, Journal for Analysis and its Applications, Vol. **22**, No. 4, pp. 751–756.

[15] M. Bendahmane, T. Lepoutre and A. Marrocco (2009), *Conservative cross diffusions and pattern formation through relaxation*, J. Maths Pures et Appl. **92**, No. 6, pp. 651–667.

[16] J. F. Blowey, J. R. King and S. Langdon (2007), *Small- and waiting-time behavior of the thin-film equation*, SIAM J. Appl. Math. **67**, No. 6, pp. 1776–1807.

[17] S. C. Brenner and L. R. Scott (2002), *The Mathematical Theory of Finite Element Methods*, Second Edition, Springer, New York.

[18] R. L. Burden and J.D. Faires (2005), *Numerical Analysis*, 8th edition, Thomson Brooks/Cole, USA.

[19] T. Cazenave (2003), *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, American Mathematical Society, Vol. **10**, New York.

[20] L. Chen, L. Hsiao and Y. Li (2003), *Strong solution to a kind of cross diffusion parabolic system*, Comm. Math. Sci. **1**, pp. 799–808.

[21] L. Chen and A. Jüngel (2004), *Analysis of a multi-dimensional parabolic population model with strong cross-diffusion*, SIAM J. Math. Anal. **36**, pp. 301–322.

[22] L. Chen and A. Jüngel (2006), *Analysis of a parabolic cross-diffusion population model without self-diffusion*, J. Diff. Eqs. **224**, pp. 39–59.

[23] P. G. Ciarlet (1978), *The Finite Element Method for Elliptic Problems*, North-Holland.

[24] P. G. Ciarlet and P. A. Raviart (1972), *General Lagrange and Hermite Interpolation in R^n with Applications to Finite Element Methods*, Archive for Rational Mechanics and Analysis, **46**, pp. 177–199.

[25] K. E. Custafson (1980), *Introduction to Partial Differential Equations and Hilbert Space Methods*, John Wiley and Sons, New York.

[26] R. Dautray and J. L. Lions (1988), *Mathematical Analysis and Numerical Methods for Science and Technology* Volume 2: *Functional and Variational Methods*, Spring-Verlag, Berlin.

[27] J. J. Dongarra, J. R. Bunch, C. B. Moler and G. W. Stewart (1979), *Linpack User's Guide*, Society for Industrial and Applied Mathematics, Philadelphia.

[28] B. Dubey, B. Das and J. Hussian (2001), *A predatorprey interaction model with self and cross-diffusion*, Ecol. Model. **141**, pp. 67–76.

[29] C. Elliott and S. Luckhaus (1991), *A generalized diffusion equation for phase separation of a multicomponent mixture with interfacial free energy*, Preprint 887, IMA, University of Minnesota.

[30] C. M. Elliott and A. M. Stuart (1993), *The Global Dynamics of Discrete Semilinear Parabolic Equations*, SIAM J. Numer. Anal. **30**, pp. 1622–1663.

[31] E. Emmrich (1999), *Discrete versions of Gronwall's lemma and their application to the numerical analysis of parabolic problems*, Preprint No. 637, Fachbereich Mathematik, TU Berlin, pp. 1–36.

[32] A. Ern and J. Guermond (2004), *Theory and Practice of Finite Elements*, Applied Mathematical Sciences, Vol. **159**, Springer, New York.

[33] G. Galiano, M. L. Garzón, and A. Jüngel (2003), *Semi-discretization in time and numerical convergence of solutions of a nonlinear cross-diffusion population model*, Numer. Math. **93**, pp. 655–673.

[34] G. Galiano, A. Jüngel and J. Velasco (2003), *A parabolic cross-diffusion system for granular materials*, SIAM J. Math. Anal. **35**, pp. 561–578.

[35] M. R. Garvie (2003), *Analysis of Reaction-Diffusion System of Lambda-Omega Type*, University of Durham, Ph.D. thesis.

[36] G. Grün and M. Rumpf (2000), *Nonnegativity preserving numerical schemes for the thin film equation*, Numer. Math. **87**, pp. 113–152.

[37] M. P. Gualdani, A. Jüngel, and G. Toscani (2003), *Exponential decay in time of solutions of the viscous quantum hydrodynamic equations*, Appl. Math. Lett. **16**, pp. 1273–1278.

[38] W. Hackbusch (1992), *Elliptic Differential Equations: Theory and Numerical Treatment*, Springer Series in Computational Mathematics, Volume **18**, Springer-Verlag, Berlin.

[39] L. Hadjadj, K. Hamdache and D. Hamroun (2008), *An existence result to a strongly coupled degenerated system arising in tumor modeling*, Abstract and Applied Analysis, Volume 2008, Article ID 239870, 19 pages (electronic).

[40] A. S. Hegazi, E. Ahmed and A. S. Elgazzar (2004), *On persistence and stability of some biological systems with cross-diffusion*, Adv. Complex Syst. **7**, No. 1, pp. 65–76.

[41] M. Imran (2001), *Numerical Analysis of a Coupled Pair of Cahn Hilliard Equations*, University of Durham, Ph.D. thesis.

[42] C. Johnson (1990), *Numerical solution of partial differential equations by the finite element method*, Cambridge University Press.

[43] Z. Khan (2006), *Granular segregation dynamics in a rotating drum*, University of Toronto, Ph.D. thesis.

[44] R. Kowalczyk (2005), *Preventing blow-up in a chemotaxis model*, J. Math. Anal. Appl. **305**, pp. 566–588.

[45] R. Kowalczyk and Z. Szymańska (2008), *On the global existence of solutions to an aggregation model*, J. Math. Anal. Appl. **343**, pp. 379–398.

[46] A. Kufner, O. John and L. S. Fucik (1977), *Function Spaces*, Noordhoff International Publishing, Leyden.

[47] A. Kurdila and M. Zabarankin (2005), *Convex Functional Analysis*, Birkhäuser Verlag, Basel, Switzerland.

[48] J. L. Lions (1969), *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod/Gauthier Villars, Paris.

[49] A. J. Lotka (1925), *Elements of physical biology*, Williams and Wilkins, Baltimore.

[50] Y. Lou and W.-M. Ni (1996), *Diffusion, self-diffusion and cross-diffusion*, J. Diff. Eqs. **131**, pp. 79–131.

[51] Y. Lou and S. Martinez (2009), *Evolution of cross-diffusion and self-diffusion*, Journal of Biological Dynamics, Vol. **3**, pp. 410–429.

[52] J. D. Murray (2003), *Mathematical Biology II: Spatial Models and Biomedical Applications*, Interdisciplinary Applied Mathematics, Vol. **18**, Third Edition, Springer, New York.

[53] R. H. Nochetto (1991), *Finite Element Methods for Parabolic Free Boundary Problems*, in Advances in Numerical Analysis, Vol. **1** (W. Light editor), Oxford University Press, Oxford, pp. 34–95.

[54] A. Okubo (1980), *Diffusion and ecological problems: mathematical models*, Biomathematics, Vol. **10**, Springer, Berlin.

[55] K. J. Painter, J. A. Sherratt (2003), *Modelling the movement of interacting cell populations*, J. Theor. Biol. **225**, pp. 327–339.

[56] P. A. Raviart (1973), *The use of numerical integration in finite element methods for solving parabolic equations*, Topics in numerical analysis, Academic Press, London, pp. 233–264.

[57] M. Renardy and R. C. Rogers (1993), *An Introduction to Partial Differential Equations*, Texts in Applied Mathematics, Vol. **13**, Springer-Verlag, New York.

[58] J. C. Robinson (2001), *Infinite-dimensional dynamical systems: an introduction to dissipative parabolic PDEs and the theory of global attractors*, Cambridge University Press.

[59] C. Schwab (1998), *p- and hp- Finite Element Methods: Theory and Applications in Solid and Fluid Mechanics*, Numerical Mathematics and Scientific Computation, Clarendon Press, Oxford.

[60] N. Shigesada, K. Kawasaki, and E. Teramoto (1979), *Spatial segregation of interacting species*, J. Theor. Biol. **79**, pp. 83–99.

[61] G. Strang and G. J. Fix (1973), *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, N. J.

[62] R. Temam (1988), *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences, Vol. **68**, Springer, New York.

[63] V. Thomée (2006), *Galerkin Finite Element Methods for Parabolic Problems*, Springer Series in Computational Mathematics, Volume **25**, Second Edition, Springer-Verlag, Berlin.

[64] M. A. Tsyganov, V. N. Biktashev, J. Brindley, A. V. Holden and G. R. Ivanitsky (2007), *Waves in systems with cross-diffusion as a new class of nonlinear waves*, Physics-Uspekhi, **50**, pp. 263–286.

[65] V. Volterra (1931), *Variations and fluctuations of a number of individuals in animal species living together*, In RN Chapman editor, *Animal Ecology*, McGraw-Hill, New York, pp. 409–448.

[66] J. Wloka (1987), *Partial Differential Equations*, Cambridge University Press, Cambridge.

[67] S. Xu (2008), *Existence of global solutions for a predator-prey model with cross-diffusion*, J. Diff. Eqs. **6**, pp. 1–14.

[68] L. Zhornitskaya and A. L. Bertozzi (2000), *Positivity preserving numerical schemes for lubrication-type equations*, SIAM J. Numer. Anal. **37**, pp. 523–555.

Appendix A

Basic and auxiliary results

A.1 Basic results

Theorem A.1.1 (Schauder's theorem) Let B be a normed space and let K be a non-empty convex compact set of B. If $f : K \to K$ is a continuous function then f has at least one fixed point (see [6] page 215).

Theorem A.1.2 (Lax-Milgram) Let H be a Hilbert space and $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ be a continuous bilinear form which is coercive, i.e., there exists $\alpha > 0$ such that

$$a(v, v) \geq \alpha \, ||v||_H^2 \quad \forall v \in H.$$

Then for every $F \in H'$ there exists a unique $u \in H$ such that

$$a(u, v) = F(v) \quad \forall v \in H.$$

Furthermore, the *a priori* estimate

$$||u||_H \leq \frac{1}{\alpha} \, ||F||_{H'}$$

holds (see, e.g., [59] page 20 and [32] page 83).

Theorem A.1.3 (generalized Lax-Milgram) Let V and W be reflexive Banach spaces. Further let $a(\cdot,\cdot) : V \times W \to \mathbb{R}$ be a continuous bilinear form such that

$$\sup_{v\in V} a(v,w) > 0 \quad \forall\, 0 \neq w \in W,$$
$$\inf_{0\neq v\in V} \sup_{0\neq w\in W} \frac{a(v,w)}{||v||_V ||w||_W} \geq \alpha,$$

where α is a positive constant. Then for every $F \in W'$ there exists a unique $u \in V$ such that

$$a(u,w) = F(w) \quad \forall w \in W.$$

Furthermore, the following *a priori* estimate holds:

$$||u||_V \leq \frac{1}{\alpha} ||F||_{W'} .$$

For a proof and applications of the theorem, see for example [59] and [32].

Theorem A.1.4 (Gilfand Triple) Let W be a Banach space continuously and densely embedded in the Hilbert space H. Then

$$W \hookrightarrow H \equiv H' \hookrightarrow W', \quad H' \text{ is dense in } W'$$

and we can write

$$\langle f, w\rangle_{W'\times W} = (f,w)_H \quad \forall\, f \in H\,,\; w \in W.$$

(See [47], page 103–105).

Definition A.1.5 (Strong convergence) Let X be a normed vector space. Then $x_n \to x$ *strongly* in X if and only if

$$||x_n - x||_X \to 0 \quad \text{as } n \to \infty\,.$$

Note that we use " $\to$ " to denote strong convergence.

Definition A.1.6 (Weak convergence) Let X be a Banach space. Then $x_n \rightharpoonup x$ *weakly* in X if and only if

$$\langle f, x_n\rangle \to \langle f, x\rangle \quad \text{as } n \to \infty \;\; \text{for every } f \in X',$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between X and X'. Note that we use " $\rightharpoonup$ " to denote weak convergence.

Definition A.1.7 (Weak-star convergence) Let X be a Banach space. Then $f_n \rightharpoonup^* f$ *weakly-star* in X' if and only if

$$\langle f_n, x\rangle \to \langle f, x\rangle \quad \text{as } n \to \infty \quad \text{for every } x \in X,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between X and X'. Note that we use " $\rightharpoonup^*$ " to denote weak-star convergence.

Theorem A.1.8 (Weak and weak-star convergence properties) Let X be a Banach space. The following statements hold:

(i) If $x_n \to x$ in X then $x_n \rightharpoonup x$ in X.

(ii) Weak limits are unique, and weakly convergent sequences are bounded.

(iii) Weak-star limits are unique, and weakly-star convergent sequences are bounded.

(iv) If $x_n \rightharpoonup x$ in X then $||x||_X \leq \liminf_{n\to\infty} ||x_n||_X$.

The proof of the above results can be found, for example, in [58] page 102–105.

Theorem A.1.9 (Weak compactness) Let X be a reflexive Banach space, $\{x_n\}$ a bounded sequence in X. Then it is possible to extract from $\{x_n\}$ a subsequence which converges weakly in X (see [26], page 289).

Theorem A.1.10 (Weak-star compactness) Let X be a separable Banach space and X' its dual. Then from every bounded sequence in X', it is possible to extract a subsequence which is weakly-star convergent in X' (see [26], page 291).

Theorem A.1.11 (Convergence) If a sequence $u_n \to u$ in $L^p(\Omega)$, $(1 \leq p < \infty)$, then there is a subsequence that converges pointwise to u almost everywhere in Ω, (see, e.g., [58] page 27).

Theorem A.1.12 (Sobolev spaces results) Let m be a non-negative integer and let $1 \leq p \leq \infty$. The Sobolev spaces $W^{m,p}(\Omega)$ equipped with the associated norms satisfy the following:

(i) $W^{m,p}(\Omega)$ is a Banach space (see [57], page 206).

(ii) $W^{m,p}(\Omega)$ is separable if $p < \infty$ (see [57], page 206).

(iii) $W^{m,p}(\Omega)$ is reflexive if $1 < p < \infty$ (see [1], page 47).

Theorem A.1.13 (Sobolev embedding results) Suppose that Ω is a bounded domain. For non-negative integers m and k such that $m \geq k$, we have

$$W^{m,q}(\Omega) \hookrightarrow W^{k,p}(\Omega)$$

whenever $1 \leq p \leq q \leq \infty$ (see, e.g., [17] page 32).

If the domain Ω has a Lipschitz boundary, there are more subtle relations among the Sobolev spaces. For instance, there are cases when $k < m$ and $p > q$ and the above embedding is satisfied. In this direction, we refer to the Sobolev embedding theorems in [1], [23] and [6].

Theorem A.1.14 (Time-Dependent spaces results) Let X be a Banach space and let $1 \leq p \leq \infty$. The Sobolev spaces $L^p(0,T;X)$ satisfy the following:

(i) $L^p(0,T;X)$ is a Banach space (see [46], page 114–116).

(ii) $L^p(0,T;X)$, $(p < \infty)$, is separable $\Leftrightarrow$ X is separable (see [46], page 118).

(iii) $L^p(0,T;X)$, $(1 < p < \infty)$, is reflexive $\Leftrightarrow$ X is reflexive (see [46], page 125).

Theorem A.1.15 (Time-Dependent spaces: embedding results) Let X, Y be Banach spaces with X continuously embedded in Y. Then

$$L^q(0,T;X) \hookrightarrow L^p(0,T;Y), \qquad 1 \leq p \leq q \leq \infty\,.$$

(See, for example, [47] page 132).

Theorem A.1.16 (Density results)

(i) Let Ω be an open bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$. Let m be a non-negative integer and $1 \leq p < \infty$. Then $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$, (see, e.g., [59] page 346).

(ii) Let X be a Banach space and $1 \leq p < \infty$. Then $C^\infty([0,T];X)$ is dense in $L^p(0,T;X)$, (see [46], page 118).

A.2 Matrices

Consider a uniform partitioning of $\Omega = (0, L)$ with mesh points $x_j = j\,h,\ j = 0, \cdots, J$. It can be easily seen that:

(i) The lumped mass matrix, $\widehat{M}$, and the stiffness matrix, K, corresponding to the finite element space S^h are of order $J+1$ and given by

$$\widehat{M} = h \begin{pmatrix} \frac{1}{2} & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad K = \frac{1}{h} \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}.$$

Therefore,

$$\widehat{M}^{-1}K = \frac{1}{h^2} \begin{pmatrix} 2 & -2 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -2 & 2 \end{pmatrix}.$$

(ii) The lumped mass matrix, $\widehat{M}$, and the stiffness matrix, K, corresponding to the finite element space S_p^h are of order J and given by

$$\widehat{M} = h \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}, \quad K = \frac{1}{h} \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & -1 & 2 & -1 \\ -1 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

Therefore, $\widehat{M}^{-1}K = \frac{1}{h}K$.

Appendix B

Fortran programs

B.1 Solver for the population model

```
C One dimensional solver for the iteration scheme of problem (P_M)
C ------------------------------------------------------------------
      PROGRAM POPULATION
      implicit none
      integer nmax,l_d_a
      parameter (l_d_a=10,nmax=257)
      integer i,n,k5,m,count,nloops,n_tot,p_loop,loop,
     .  i_max,info,nL,nR,j,k,i1,i2,even_odd,index,lda,
     .  ml,mu,nsub,msub,ipvt(2*nmax),job,i_count,fix
      double precision u(1:nmax,1:2),unm(1:nmax,1:2),un(1:nmax,1:2),
     .  a(1:2*nmax,1:2*nmax),B(1:2*nmax),abd(l_d_a,1:2*nmax),
     .  mue(1:2,1:2),gamma(1:2),h,h2,time,len,t,tol,eps,epse,
     .  mb,me,dd,tau,diff,c,dF,d2F,Cross,v_L,v_R,mult,pi
      character*30 datafile1,datafile2
      character*1 number1
      character*2 number2
      character*3 lettert1,lettert2,number3
      character*4 number4
C
C Declare the values of some characters (for printing results)
C
      lettert1='t1_'
      lettert2='t2_'
C
C Input some important variables from population.dat
C
C len  = length of $Omega$
```

```
C m    = number of space steps
C t    = final time
C n    = number of time steps
C tol  = tolerance level for iterative loop
C k5   = number of prints
C
      open(1,status='old',file='population.dat')
        read(1,*) len
        read(1,*) m
        read(1,*) t
        read(1,*) n
        read(1,*) tol
        read(1,*) k5
      close(1)
C
      if (mod(n,k5).ne.0) then
        print *, 'Enter k5 - must be a factor of n'
        read(5,*) k5
      end if
C
C Define some variables
C
C tau    = time step
C h      = space step
C count  = current time step
C time   = time level (time = 0, tau, 2tau, ... , t)
C nloops = number of iterations for each time step
C n_tot  = total number of iterations for all time steps combined
C eps    = the regularization parameter
C mb     = the constant 'M' in the truncated problem
C
      tau=t/real(n)
      print *,m,n,tol
      h=len/real(m)
      h2=h**2
      count=0
      time = 0.0D0
      nloops=0
      n_tot=0
      pi=3.1415926535897932385
      eps=1.0D-9
      epse=1.0D+00/eps
      mb=1.0D+00
C     me=epse
C Choose the parameters of the model
```

```
      dd=1.0D+00
      mue(1,1)=1.0D+00
      mue(1,2)=1.0D+00
      mue(2,1)=1.0D+00
      mue(2,2)=1.0D+00
      gamma(1)=1.0D+00
      gamma(2)=1.0D+00
C Define and print the initial functions
      nL=0
      nR=m
      open(10,status='new',file='t1_0.dat')
      open(20,status='new',file='t2_0.dat')
      do 10 i=1,nR-nL+1
         u(i,1)=-0.2D+00*(real(i-1)*h)+1
         u(i,2)=1.0D+00
       write(10,*) u(i,1)
       write(20,*) u(i,2)
 10   continue
       close(10)
       close(20)
C
C For a band matrix, we define:
C ml  = number of diagonals below the main diagonal
C mu  = number of diagonals above the main diagonal
C
         ml=3
         mu=3
         lda=2*ml+mu+1
C
C Set U^{1,0} = U^{0} , V^{1,0} = V^{0}
C     un(.,1) = U^{n,k}, unm(.,1) = U^{n,k-1}
C     un(.,2) = V^{n,k}, unm(.,2) = V^{n,k-1}
C
      do 80 i=1,nR-nL+1
        unm(i,1)=u(i,1)
        unm(i,2)=u(i,2)
        un(i,1)=u(i,1)
        un(i,2)=u(i,2)
 80   continue
C
C We start the print loop to print the results every n/k5 time steps
C
        do 700 p_loop=1,k5
C
C We define the solve loop for each print
```

```
C
          do 800 loop=1,n/k5
C
C This is the beginning of the iterative loop
C Given U^{n-1}, V^{n-1}, U^{n,k-1} and V^{n,k-1} find U^{n,k} and V^{n,k}
C
 200        nloops=nloops+1
            job=0
C We now construct the matrix of the linear system
            do 103 i=1,2*(nR-nL+1)
              B(i)=0.0
            do 102 j=1,2*(nR-nL+1)
              A(i,j)=0.0
 102          continue
            do 104 j=1,l_d_a
              abd(j,i) = 0.0
 104          continue
 103          continue
            do 100 i=1,2*(nR-nL+1)
              if (i.le.(nR-nL+1)) then
                j=2
                k=1
                me=mb
              else
                j=1
                k=2
                me=epse
              end if
              i1=mod(i-1,nR-nL+1)+1

              if (i.le.nR-nL+1) then
                even_odd=1
              else
                even_odd=0
              end if
              if (i.gt.(nR-nL+1)) then
                fix=-2
              else
                fix=0
              end if
              index=2*i1-even_odd
C
C We define the first equation of the iterative algorithm in terms of the
C nodal values
C
```

```
      if (i.le.(nR-nL+1)) then
C The row corresponding to the first nodal value of U^{n,k}
         if (i1.eq.1) then
           mult=2.0D+00
           v_L=0.0D+00
           v_R=mult*C(un(i1,k),un(i1+1,k),eps,me,dd)*(tau/h2)

           A(index,index)=1.0D+00+(v_L+v_R)-(tau*gamma(k))
     .         +(tau*mue(k,k)*(1.0D+00/d2F(unm(i1,k),eps,me)))
           A(index,index+2)=A(index,index+2)-v_R

           v_R=mult*(tau/h2)*Cross(un(i1,k),un(i1+1,k),eps,me)
           A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
           A(index,index+3+fix)=A(index,index+3+fix)-v_R

           B(index)=(unm(i1,k)-tau*(mue(k,j)
     .                  *(1.0D+00/d2F(un(i1,k),eps,me))
     .                     *(1.0D+00/d2F(unm(i1,j),eps,epse))))
        else
C The row corresponding to the last nodal value of U^{n,k}
          if (i1.eq.nR-nL+1) then
            mult=2.0D+00
            v_R=0.0D+00
            v_L=mult*C(un(i1-1,k),un(i1,k),eps,me,dd)*(tau/h2)

            A(index,index)=1.0D+00+(v_L+v_R)-(tau*gamma(k))
     .         +(tau*mue(k,k)*(1.0D+00/d2F(unm(i1,k),eps,me)))
            A(index,index-2)=A(index,index-2)-v_L

            v_L=mult*(tau/h2)*Cross(un(i1-1,k),un(i1,k),eps,me)
            A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L

            B(index)=(unm(i1,k)-tau*(mue(k,j)
     .                 *(1.0D+00/d2F(un(i1,k),eps,me))
     .                      *(1.0D+00/d2F(unm(i1,j),eps,epse))))
          else
C The rows corresponding to the rest of the nodal values of U^{n,k}
            v_L=C(un(i1-1,k),un(i1,k),eps,me,dd)*(tau/h2)
            v_R=C(un(i1,k),un(i1+1,k),eps,me,dd)*(tau/h2)
            A(index,index)=1.0D+00+(v_L+v_R)-(tau*gamma(k))
     .         +(tau*mue(k,k)*(1.0D+00/d2F(unm(i1,k),eps,me)))
            A(index,index-2)=A(index,index-2)-v_L
            A(index,index+2)=A(index,index+2)-v_R
```

```
            v_L=(tau/h2)*Cross(un(i1-1,k),un(i1,k),eps,me)
            v_R=(tau/h2)*Cross(un(i1,k),un(i1+1,k),eps,me)
            A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L
            A(index,index+3+fix)=A(index,index+3+fix)-v_R

            B(index)=(unm(i1,k)-tau*(mue(k,j)
     .                     *(1.0D+00/d2F(un(i1,k),eps,me))
     .                        *(1.0D+00/d2F(unm(i1,j),eps,epse))))

           end if
          end if
C
C We define the second equation of the iterative algorithm in terms of the
C nodal values
C
           else
C The row corresponding to the first nodal value of V^{n,k}
            if (i1.eq.1) then
            mult=2.0D+00
            v_L=0.0D+00
            v_R=mult*C(un(i1,k),un(i1+1,k),eps,me,dd)*(tau/h2)
            A(index,index)=1.0D+00+(v_L+v_R)-(tau*gamma(k))
            A(index,index+2)=A(index,index+2)-v_R

            v_R=mult*(tau/h2)*Cross(un(i1,k),un(i1+1,k),eps,me)
            A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
            A(index,index+3+fix)=A(index,index+3+fix)-v_R

              B(index)=(unm(i1,k)-tau*(mue(k,k)
     .                  *(1.0D+00/d2F(un(i1,k),eps,me))
     .                      *(1.0D+00/d2F(unm(i1,k),eps,me))
     .                        +mue(k,j)*(1.0D+00/d2F(un(i1,k),eps,me))
     .                            *(1.0D+00/d2F(unm(i1,j),eps,mb))))

         else
C The row corresponding to the last nodal value of V^{n,k}
          if (i1.eq.nR-nL+1) then
            mult=2.0D+00
            v_R=0.0D+00
            v_L=mult*C(un(i1-1,k),un(i1,k),eps,me,dd)*(tau/h2)

            A(index,index)=1.0D+00+(v_L+v_R)-(tau*gamma(k))
            A(index,index-2)=A(index,index-2)-v_L
```

```
            v_L=mult*(tau/h2)*Cross(un(i1-1,k),un(i1,k),eps,me)
            A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L

              B(index)=(unm(i1,k)-tau*(mue(k,k)
     .                 *(1.0D+00/d2F(un(i1,k),eps,me))
     .                     *(1.0D+00/d2F(unm(i1,k),eps,me))
     .                       +mue(k,j)*(1.0D+00/d2F(un(i1,k),eps,me))
     .                          *(1.0D+00/d2F(unm(i1,j),eps,mb))))

          else
C The rows corresponding to the rest of the nodal values of V^{n,k}
            v_L=C(un(i1-1,k),un(i1,k),eps,me,dd)*(tau/h2)
            v_R=C(un(i1,k),un(i1+1,k),eps,me,dd)*(tau/h2)
            A(index,index)=1.0D+00+(v_L+v_R)-(tau*gamma(k))
            A(index,index-2)=A(index,index-2)-v_L
            A(index,index+2)=A(index,index+2)-v_R

            v_L=(tau/h2)*Cross(un(i1-1,k),un(i1,k),eps,me)
            v_R=(tau/h2)*Cross(un(i1,k),un(i1+1,k),eps,me)
            A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L
            A(index,index+3+fix)=A(index,index+3+fix)-v_R

              B(index)=(unm(i1,k)-tau*(mue(k,k)
     .                  *(1.0D+00/d2F(un(i1,k),eps,me))
     .                    *(1.0D+00/d2F(unm(i1,k),eps,me))
     .                      +mue(k,j)*(1.0D+00/d2F(un(i1,k),eps,me))
     .                        *(1.0D+00/d2F(unm(i1,j),eps,mb))))
          end if
          end if
        end if

 100        continue
C
C We define the matrix abd which contains the matrix A in band storage
C
         nsub=2*(nR-nL+1)
                    msub = ml + mu + 1
                    do 201 j = 1, nsub
                       i1 = max0(1, j-mu)
                       i2 = min0(nsub, j+ml)
                       do 101 i = i1, i2
                          k = i - j + msub
                          abd(k,j) = A(i,j)
```

```
 101                  continue
 201               continue
C
C We now call the subroutine DGBFA to factor the band matrix A using Gaussian
C elimination. Then we call the subroutine DGBSL to solve the band system
C A*X=B using the factors computed by DGBFA. The subroutines DGBFA and DGBSL
C can be found in reference [27].
C
          CALL DGBFA(abd,lda,nsub,ml,mu,ipvt,info)
          CALL DGBSL(abd,lda,nsub,ml,mu,ipvt,b,job)
C
C Check if diff = max {\|U^{n,k}-U^{n,k-1}\|, \|V^{n,k}-V^{n,k-1}\|} < tol
C Reset unm(.,1) = U^{n,k} and unm(.,2) = V^{n,k} for the next time level
C
          diff=0.0D0
          i_max=0
          do 260 i=1,2*(nR-nL+1)
            if (mod(i-1,2).eq.0) then
              i_count=(i-1)/2+1
              if (diff.lt.dabs(un(i_count,1)-B(i))) then
                diff=dabs(un(i_count,1)-B(i))
                i_max=i
              end if
              un(i_count,1)=B(i)
            else
              i_count=i/2
              if (diff.lt.dabs(un(i_count,2)-B(i))) then
                diff=dabs(un(i_count,2)-B(i))
                i_max=i
              end if
              un(i_count,2)=B(i)
            end if
          B(i)=0.0
 260      continue
C and if diff > tol then repeat iterative loop
          if (mod(nloops,10).eq.0) print *,nloops,diff,i_max
          write(21,*) diff
          if (diff.gt.tol) goto 200
C
C This is the end of the iterative loop
C
          if (mod(loop-1,10).eq.0) then
            print *,p_loop, loop, nloops, diff
          end if
C
```

```
C Set U^{n-1}=U^{n,k} and V^{n-1}=V^{n,k}
C
          do 270 i=1,nR-nL+1
            unm(i,1)=un(i,1)
            unm(i,2)=un(i,2)
 270      continue
C
C Store total number of iterations in n_tot
C
          n_tot=n_tot+nloops
          nloops=0
          if (mod(loop,1000).eq.0) print *,n_tot
          time=time+tau
          count=count+1
 800    continue
C
C Printing the results:
C We first identify a location for the output according to the size of p_loop
C
        if (p_loop.le.9) then
          write(number1,910) p_loop
          datafile1 =lettert1//number1//'.dat'
          datafile2 =lettert2//number1//'.dat'
        else
          if (p_loop.le.99) then
            write(number2,920) p_loop
            datafile1 =lettert1//number2//'.dat'
            datafile2 =lettert2//number2//'.dat'
          else
            if (p_loop.le.999) then
              write(number3,930) p_loop
              datafile1 =lettert1//number3//'.dat'
              datafile2 =lettert2//number3//'.dat'
            else
              write(number4,940) p_loop
              datafile1 =lettert1//number4//'.dat'
              datafile2 =lettert2//number4//'.dat'
            end if
          end if
        endif
C
C We store the solutions in the appropriate files (U^{p_loop} and V^{p_loop}
C will be stored in the data files t1_(p_loop) and t2_(p_loop) respectively)
C
        open(1,status='new',file=datafile1)
```

```
        open(2,status='new',file=datafile2)
          do 640 i=1,nR-nL+1
              if (mod(i-1,1).eq.0) then
              write(1,*) un(i,1)
              write(2,*) un(i,2)
              end if
 640      continue
        close(1)
        close(2)

 700  continue
C      print *, n_tot

 910  format(i1)
 920  format(i2)
 930  format(i3)
 940  format(i4)
      stop
      end
C
C ---------------------------------------------------
C
C This function calculates the diffusion coefficient
C
      DOUBLE PRECISION FUNCTION C(uL,uR,eps,me,dd)
      implicit none
      double precision uL,uR,eps,me,dd,dF,d2F
      if (abs(uR-uL).lt.1D-12) then
        c=dd+1.0D+00/d2F(uR,eps,me)
      else
        c=dd+(uR-uL)/(dF(uR,eps,me)-dF(uL,eps,me))
      endif
      end
C
C This function calculates the cross diffusion coefficient
C
      DOUBLE PRECISION FUNCTION Cross(u1L,u1R,eps,me)
      implicit none
      double precision u1L,u1R,eps,me,dF,d2F
      if (abs(u1R-u1L).lt.1D-12) then
          Cross=1.0D+00/(d2F(u1R,eps,me))
      else
        Cross=(u1R-u1L)/(dF(u1R,eps,me)-dF(u1L,eps,me))
      endif
      end
```

```
C
C The first derivative of the function F ( or G )
C
      DOUBLE PRECISION FUNCTION dF(s,eps,me)
      implicit none
      double precision s,eps,me
      if (s.lt.eps) then
        dF=dlog(eps)+(s-eps)/eps
      else
        if (s.ge.me) then
          dF=dlog(me)+(s-me)/me
          else
            dF=dlog(s)
        end if
      end if
      end
C
C The second derivative of the function F ( or G )
C
      DOUBLE PRECISION FUNCTION d2F(s,eps,me)
      implicit none
      double precision s,eps,me
      if (s.lt.eps) then
        d2F=1.0D+00/eps
      else
        if (s.ge.me) then
          d2F=1.0D+00/me
          else
            d2F=1.0D+00/s
        end if
      end if
      end
C ---------------------------------------------------
C ---------------------------------------------------
```

B.2 Solver for the axial segregation model

B.2.1 With Neumann boundary conditions

```
C Program to solve the iteration scheme of problem (Q)
C with Neumann boundary conditions
C ----------------------------------------------------
      PROGRAM AXIAL
      implicit none
      integer nmax,l_d_a
      parameter (l_d_a=10,nmax=513)
      integer i,n,k5,m,count,nloops,n_tot,p_loop,loop,
     .  i_max,info,nL,nR,j,k,i1,i2,even_odd,index,lda,
     .  ml,mu,nsub,msub,ipvt(2*nmax),job,i_count,fix
      double precision u(1:nmax,1:2),unm(1:nmax,1:2),un(1:nmax,1:2),
     .  a(1:2*nmax,1:2*nmax),B(1:2*nmax),abd(l_d_a,1:2*nmax),len,t,
     .  tol,eps,lambda,mue,theta,tau,h,h2,time,pi,diff,dP,d2P,rho,
     .  Cross,v_L,v_R,mult
      character*30 datafile1,datafile2
      character*1 number1
      character*2 number2
      character*3 lettert1,lettert2,number3
      character*4 number4
C
C Declare the values of some characters (for printing results)
C
      lettert1='t1_'
      lettert2='t2_'
C
C Input some important variables from population.dat
C
C len  = length of $Omega$
C m    = number of space steps
C t    = final time
C n    = number of time steps
C tol  = tolerance level for iterative loop
C k5   = number of prints
C
      open(1,status='old',file='axial.dat')
        read(1,*) len
        read(1,*) m
        read(1,*) t
        read(1,*) n
        read(1,*) tol
        read(1,*) k5
```

```
      close(1)

C
      if (mod(n,k5).ne.0) then
        print *, 'Enter k5 - must be a factor of n'
        read(5,*) k5
      end if
C
C Define some variables
C
C tau    = time step
C h      = space step
C count  = current time step
C time   = time level (time = 0, tau, 2tau, ... , t)
C nloops = number of iterations for each time step
C n_tot  = total number of iterations for all time steps combined
C eps    = the regularization parameter
C
      tau=t/real(n)
      print *,m,n,tol
      h=len/real(m)
      h2=h**2
      count=0
      time = 0.0D0
      nloops=0
      n_tot=0
      pi=3.1415926535897932385
      eps=1.0D-9
C Choose the parameters of the model
      lambda=2.0D+00
      rho=2.0D+00
      mue=3.0D+00
      theta=1.0D+00
C Define and print the initial functions
      nL=0
      nR=m
      open(10,status='new',file='t1_0.dat')
      open(20,status='new',file='t2_0.dat')
      do 10 i=1,nR-nL+1
         u(i,1)=0.80D+00*cos(4.0D+00*real(i-1)*h*pi/len)
         u(i,2)=0.0D+00
       write(10,*) u(i,1)
       write(20,*) u(i,2)
 10   continue
      close(10)
```

```
      close(20)
C
C For a band matrix, we define:
C ml  = number of diagonals below the main diagonal
C mu  = number of diagonals above the main diagonal
C
         ml=3
         mu=3
         lda=2*ml+mu+1
C
C Set W^{1,0} = W^{0} , Z^{1,0} = Z^{0}
C     un(.,1) = W^{n,k}, unm(.,1) = W^{n,k-1}
C     un(.,2) = Z^{n,k}, unm(.,2) = Z^{n,k-1}
C
      do 80 i=1,nR-nL+1
        unm(i,1)=u(i,1)
        unm(i,2)=u(i,2)
        un(i,1)=u(i,1)
        un(i,2)=u(i,2)
 80   continue
C
C We start the print loop to print the results every n/k5 time steps
C
        do 700 p_loop=1,k5
C
C We define the solve loop for each print
C
          do 800 loop=1,n/k5
C
C This is the beginning of the iterative loop
C Given W^{n-1}, Z^{n-1} and W^{n,k-1} find W^{n,k} and Z^{n,k}
C
 200        nloops=nloops+1
            job=0
C We now construct the matrix of the linear system
            do 103 i=1,2*(nR-nL+1)
              B(i)=0.0
            do 102 j=1,2*(nR-nL+1)
              A(i,j)=0.0
 102          continue
            do 104 j=1,l_d_a
              abd(j,i) = 0.0
 104          continue
 103          continue
            do 100 i=1,2*(nR-nL+1)
```

```
              if (i.le.(nR-nL+1)) then
                j=2
                k=1
              else
                j=1
                k=2
              end if
              i1=mod(i-1,nR-nL+1)+1

              if (i.le.nR-nL+1) then
                even_odd=1
              else
                even_odd=0
              end if
              if (i.gt.(nR-nL+1)) then
                fix=-2
              else
                fix=0
              end if
              index=2*i1-even_odd
C
C We define the first equation of the iterative algorithm in terms of the
C nodal values
C
      if (i.le.(nR-nL+1)) then
C The row corresponding to the first nodal value of W^{n,k}
         if (i1.eq.1) then
           mult=2.0D+00
           v_L=0.0D+00
           v_R=mult*rho*(tau/h2)
           A(index,index)=1.0D+00+(v_L+v_R)
           A(index,index+2)=A(index,index+2)-v_R

           v_R=mult*(-lambda)*(tau/h2)
     .             *Cross(un(i1,k),un(i1+1,k),eps,lambda)
           A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
           A(index,index+3+fix)=A(index,index+3+fix)-v_R

           B(index)=(unm(i1,k))
        else
C The row corresponding to the last nodal value of W^{n,k}
          if (i1.eq.nR-nL+1) then
            mult=2.0D+00
            v_R=0.0D+00
            v_L=mult*rho*(tau/h2)
```

```
            A(index,index)=1.0D+00+(v_L+v_R)
            A(index,index-2)=A(index,index-2)-v_L

            v_L=mult*(-lambda)*(tau/h2)
     .               *Cross(un(i1-1,k),un(i1,k),eps,lambda)
            A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L

            B(index)=(unm(i1,k))

          else
C The rows corresponding to the rest of the nodal values of W^{n,k}
            v_L=rho*(tau/h2)
            v_R=rho*(tau/h2)
            A(index,index)=1.0D+00+(v_L+v_R)
            A(index,index-2)=A(index,index-2)-v_L
            A(index,index+2)=A(index,index+2)-v_R

            v_L=(tau/h2)*(-lambda)*Cross(un(i1-1,k),un(i1,k),eps,lambda)
            v_R=(tau/h2)*(-lambda)*Cross(un(i1,k),un(i1+1,k),eps,lambda)
            A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L
            A(index,index+3+fix)=A(index,index+3+fix)-v_R

            B(index)=(unm(i1,k))
           end if
          end if
C
C We define the second equation of the iterative algorithm in terms of the
C nodal values
C
           else
C The row corresponding to the first nodal value of Z^{n,k}
            if (i1.eq.1) then
             mult=2.0D+00
             v_L=0.0D+00
             v_R=mult*(tau/h2)
             A(index,index)=1.0D+00+tau+(v_L+v_R)
             A(index,index+2)=A(index,index+2)-v_R

             v_R=mult*(tau/h2)*lambda
             A(index,index+1+fix)=A(index,index+1+fix)
     .                               +(v_L+v_R-tau*mue*theta)
             A(index,index+3+fix)=A(index,index+3+fix)-v_R
```

```
             B(index)=(unm(i1,k)+tau*mue*(1.0D+00-theta)*unm(i1,j))

         else
C The row corresponding to the last nodal value of Z^{n,k}
          if (i1.eq.nR-nL+1) then
            mult=2.0D+00
            v_R=0.0D+00
            v_L=mult*(tau/h2)
            A(index,index)=1.0D+00+tau+(v_L+v_R)
            A(index,index-2)=A(index,index-2)-v_L

            v_L=mult*(tau/h2)*lambda
            A(index,index+1+fix)=A(index,index+1+fix)
     .                               +(v_L+v_R-tau*mue*theta)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L

            B(index)=(unm(i1,k)+tau*mue*(1.0D+00-theta)*unm(i1,j))
          else
C The rows corresponding to the rest of the nodal values of Z^{n,k}
            v_L=tau/h2
            v_R=tau/h2
            A(index,index)=1.0D+00+tau+(v_L+v_R)
            A(index,index-2)=A(index,index-2)-v_L
            A(index,index+2)=A(index,index+2)-v_R

            v_L=(tau/h2)*lambda
            v_R=(tau/h2)*lambda
            A(index,index+1+fix)=A(index,index+1+fix)
     .                               +(v_L+v_R-tau*mue*theta)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L
            A(index,index+3+fix)=A(index,index+3+fix)-v_R

            B(index)=(unm(i1,k)+tau*mue*(1.0D+00-theta)*unm(i1,j))
          end if
         end if
        end if

 100        continue
C
C We define the matrix abd which contains the matrix A in band storage
C
         nsub=2*(nR-nL+1)
                   msub = ml + mu + 1
                   do 201 j = 1, nsub
```

```
                      i1 = max0(1, j-mu)
                      i2 = min0(nsub, j+ml)
                      do 101 i = i1, i2
                         k = i - j + msub
                         abd(k,j) = A(i,j)
 101                  continue
 201               continue
C
C We now call the subroutine DGBFA to factor the band matrix A using Gaussian
C elimination. Then we call the subroutine DGBSL to solve the band system
C A*X=B using the factors computed by DGBFA. The subroutines DGBFA and DGBSL
C can be found in reference [27].
C
          CALL DGBFA(abd,lda,nsub,ml,mu,ipvt,info)
          CALL DGBSL(abd,lda,nsub,ml,mu,ipvt,b,job)
C
C Check if diff = max {\|W^{n,k}-W^{n,k-1}\|, \|Z^{n,k}-Z^{n,k-1}\|} < tol
C Reset unm(.,1) = W^{n,k} and unm(.,2) = Z^{n,k} for the next time level
C
          diff=0.0D0
          i_max=0
          do 260 i=1,2*(nR-nL+1)
            if (mod(i-1,2).eq.0) then
              i_count=(i-1)/2+1
              if (diff.lt.dabs(un(i_count,1)-B(i))) then
                diff=dabs(un(i_count,1)-B(i))
                i_max=i
              end if
              un(i_count,1)=B(i)
            else
              i_count=i/2
              if (diff.lt.dabs(un(i_count,2)-B(i))) then
                diff=dabs(un(i_count,2)-B(i))
                i_max=i
              end if
              un(i_count,2)=B(i)
            end if
          B(i)=0.0
 260      continue
C and if diff > tol then repeat iterative loop
          if (mod(nloops,10).eq.0) print *,nloops,diff,i_max
          write(21,*) diff
          if (diff.gt.tol) goto 200
C
C This is the end of the iterative loop
```

```
C
          if (mod(loop-1,10).eq.0) then
            print *,p_loop, loop, nloops, diff
          end if
C
C Set W^{n-1}=W^{n,k} and Z^{n-1}=Z^{n,k}
C
          do 270 i=1,nR-nL+1
            unm(i,1)=un(i,1)
            unm(i,2)=un(i,2)
 270      continue
C
C Store total number of iterations in n_tot
C
          n_tot=n_tot+nloops
          nloops=0
          if (mod(loop,1000).eq.0) print *,n_tot
          time=time+tau
          count=count+1
 800    continue
C
C Printing the results:
C We first identify a location for the output according to the size of p_loop
C
        if (p_loop.le.9) then
          write(number1,910) p_loop
          datafile1 =lettert1//number1//'.dat'
          datafile2 =lettert2//number1//'.dat'
        else
          if (p_loop.le.99) then
            write(number2,920) p_loop
            datafile1 =lettert1//number2//'.dat'
            datafile2 =lettert2//number2//'.dat'
          else
            if (p_loop.le.999) then
              write(number3,930) p_loop
              datafile1 =lettert1//number3//'.dat'
              datafile2 =lettert2//number3//'.dat'
            else
              write(number4,940) p_loop
              datafile1 =lettert1//number4//'.dat'
              datafile2 =lettert2//number4//'.dat'
            end if
          end if
        endif
```

```
C
C We store the solutions in the appropriate files (W^{p_loop} and Z^{p_loop}
C will be stored in the data files t1_(p_loop) and t2_(p_loop) respectively)
C
       open(1,status='new',file=datafile1)
       open(2,status='new',file=datafile2)
          do 640 i=1,nR-nL+1
              if (mod(i-1,1).eq.0) then
              write(1,*) un(i,1)
              write(2,*) un(i,2)
              end if
 640      continue
       close(1)
      close(2)

 700  continue
C      print *, n_tot

 910  format(i1)
 920  format(i2)
 930  format(i3)
 940  format(i4)
      stop
      end
C
C ---------------------------------------------------
C
C This function calculates the cross diffusion coefficient
C
      DOUBLE PRECISION FUNCTION Cross(u1L,u1R,eps,lambda)
      implicit none
      double precision u1L,u1R,eps,lambda,dP,d2P
      if (abs(u1R-u1L).lt.1D-12) then
          Cross=1.0D+00/(d2P(u1R,eps,lambda))
      else
        Cross=(u1R-u1L)/(dP(u1R,eps,lambda)-dP(u1L,eps,lambda))
      endif
      end
C
C The first derivative of the function Phi
C
      DOUBLE PRECISION FUNCTION dP(s,eps,lambda)
      implicit none
      double precision s,eps,lambda
      if (s.ge.(1.0D+00-eps)) then
```

```
          dP=(lambda/2.0D+00)*(1.0D+00+dlog(1.0D+00+s)
     .              -(1.0D+00/eps)*(1.0D+00-s)-dlog(eps))
        else
          if (s.lt.(eps-1.0D+00)) then
            dP=(lambda/2.0D+00)*(dlog(eps)+(1.0D+00/eps)*(1.0D+00+s)
     .                  -dlog(1.0D+00-s)-1.0D+00)
            else
              dP=(lambda/2.0D+00)*(dlog(1.0D+00+s)-dlog(1.0D+00-s))
          end if
        end if
        end
C
C The second derivative of the function Phi
C
        DOUBLE PRECISION FUNCTION d2P(s,eps,lambda)
        implicit none
        double precision s,eps,lambda
        if (s.ge.(1.0D+00-eps)) then
          d2P=(lambda/2.0D+00)*(1.0D+00/(1.0D+00+s)+1.0D+00/eps)
        else
          if (s.lt.(eps-1.0D+00)) then
            d2P=(lambda/2.0D+00)*(1.0D+00/(1.0D+00-s)+1.0D+00/eps)
            else
              d2P=lambda/(1.0D+00-(s**2))
          end if
        end if
        end
C ---------------------------------------------------
C ---------------------------------------------------
```

B.2.2 With periodic boundary conditions

```
C Program to solve the iteration scheme of problem (Q)
C with periodic boundary conditions
C ---------------------------------------------------
        PROGRAM AXIAL
        implicit none
        integer nmax
        parameter (nmax=512)
        integer i,n,k5,m,count,nloops,n_tot,p_loop,loop,
     .    i_max,info,nL,nR,j,k,i1,i2,even_odd,index,lda,
     .    ml,mu,ipvt(2*nmax),job,i_count,fix
        double precision u(1:nmax,1:2),unm(1:nmax,1:2),un(1:nmax,1:2),
     .    a(2*nmax,1:2*nmax),B(1:2*nmax),len,t,tol,eps,lambda,mue,
     .    theta,tau,h,h2,time,pi,diff,dP,d2P,rho,Cross,v_L,v_R
```

```
      character*30 datafile1,datafile2
      character*1 number1
      character*2 number2
      character*3 lettert1,lettert2,number3
      character*4 number4
C
C Declare the values of some characters (for printing results)
C
      lettert1='t1_'
      lettert2='t2_'
C
C Input some important variables from population.dat
C
C len  = length of $Omega$
C m    = number of space steps
C t    = final time
C n    = number of time steps
C tol  = tolerance level for iterative loop
C k5   = number of prints
C
      open(1,status='old',file='axial.dat')
        read(1,*) len
        read(1,*) m
        read(1,*) t
        read(1,*) n
        read(1,*) tol
        read(1,*) k5
      close(1)

C
      if (mod(n,k5).ne.0) then
        print *, 'Enter k5 - must be a factor of n'
        read(5,*) k5
      end if
C
C Define some variables
C
C tau    = time step
C h      = space step
C count  = current time step
C time   = time level (time = 0, tau, 2tau, ... , t)
C nloops = number of iterations for each time step
C n_tot  = total number of iterations for all time steps combined
C eps    = the regularization parameter
C
```

```
      tau=t/real(n)
      print *,m,n,tol
      h=len/real(m)
      h2=h**2
      count=0
      time = 0.0D0
      nloops=0
      n_tot=0
      pi=3.1415926535897932385
      eps=1.0D-9
C Choose the parameters of the model
      lambda=2.0D+00
      rho=2.0D+00
      mue=3.0D+00
      theta=1.0D+00
C Define and print the initial functions
      nL=0
      nR=m-1
      open(10,status='new',file='t1_0.dat')
      open(20,status='new',file='t2_0.dat')
      do 10 i=1,nR-nL+1
         u(i,1)=0.80D+00*cos(4.0D+00*real(i-1)*h*pi/len)
         u(i,2)=0.0D+00
       write(10,*) u(i,1)
       write(20,*) u(i,2)
 10   continue
C the following two lines are added to pint w(L,.):=w(0,.) and z(L,.):=z(0,.)
       write(10,*) u(1,1)
       write(20,*) u(1,2)
      close(10)
      close(20)
C Define the dimension of the matrix of the linear system
         lda=2*(nR-nL+1)
         print *, nL,nR,nR-nL+1,lda
C
C Set W^{1,0} = W^{0} , Z^{1,0} = Z^{0}
C     un(.,1) = W^{n,k}, unm(.,1) = W^{n,k-1}
C     un(.,2) = Z^{n,k}, unm(.,2) = Z^{n,k-1}
C
      do 80 i=1,nR-nL+1
        unm(i,1)=u(i,1)
        unm(i,2)=u(i,2)
        un(i,1)=u(i,1)
        un(i,2)=u(i,2)
 80   continue
```

```
C
C We start the print loop to print the results every n/k5 time steps
C
        do 700 p_loop=1,k5
C
C We define the solve loop for each print
C
          do 800 loop=1,n/k5
C
C This is the beginning of the iterative loop
C Given W^{n-1}, Z^{n-1} and W^{n,k-1} find W^{n,k} and Z^{n,k}
C
 200        nloops=nloops+1
            job=0
C We now construct the matrix of the linear system
            do 103 i=1,2*(nR-nL+1)
              B(i)=0.0
            do 102 j=1,2*(nR-nL+1)
              A(i,j)=0.0
 102          continue
 103          continue
            do 100 i=1,2*(nR-nL+1)
              if (i.le.(nR-nL+1)) then
                j=2
                k=1
              else
                j=1
                k=2
              end if
              i1=mod(i-1,nR-nL+1)+1

              if (i.le.nR-nL+1) then
                even_odd=1
              else
                even_odd=0
              end if
              if (i.gt.(nR-nL+1)) then
                fix=-2
              else
                fix=0
              end if
              index=2*i1-even_odd
C
C We define the first equation of the iterative algorithm in terms of the
C nodal values
```

```
C
      if (i.le.(nR-nL+1)) then
C The row corresponding to the first nodal value of W^{n,k}
         if (i1.eq.1) then
           v_L=rho*(tau/h2)
           v_R=rho*(tau/h2)
           A(index,index)=1.0D+00+(v_L+v_R)
           A(index,index+2)=A(index,index+2)-v_L
           A(index,m+m-1)=A(index,m+m-1)-v_R
           v_L=(-lambda)*(tau/h2)
     .             *Cross(un(i1,k),un(i1+1,k),eps,lambda)
           v_R=(-lambda)*(tau/h2)
     .             *Cross(un(m,k),un(i1,k),eps,lambda)
           A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
           A(index,index+3+fix)=A(index,index+3+fix)-v_L
           A(index,m+m)=A(index,m+m)-v_R

           B(index)=(unm(i1,k))
        else
C The row corresponding to the "pre-last" nodal value of W^{n,k}
          if (i1.eq.nR-nL+1) then
            v_R=rho*(tau/h2)
            v_L=rho*(tau/h2)
            A(index,index)=1.0D+00+(v_L+v_R)
            A(index,index-2)=A(index,index-2)-v_L
            A(index,1)=A(index,1)-v_R

            v_L=(-lambda)*(tau/h2)
     .              *Cross(un(i1-1,k),un(i1,k),eps,lambda)
            v_R=(-lambda)*(tau/h2)
     .              *Cross(un(i1,k),un(1,k),eps,lambda)
            A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L
            A(index,2)=A(index,2)-v_R

            B(index)=(unm(i1,k))

          else
C The rows corresponding to the rest of the nodal values of W^{n,k}
            v_L=rho*(tau/h2)
            v_R=rho*(tau/h2)
            A(index,index)=1.0D+00+(v_L+v_R)
            A(index,index-2)=A(index,index-2)-v_L
            A(index,index+2)=A(index,index+2)-v_R
```

```
            v_L=(tau/h2)*(-lambda)*Cross(un(i1-1,k),un(i1,k),eps,lambda)
            v_R=(tau/h2)*(-lambda)*Cross(un(i1,k),un(i1+1,k),eps,lambda)
            A(index,index+1+fix)=A(index,index+1+fix)+(v_L+v_R)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L
            A(index,index+3+fix)=A(index,index+3+fix)-v_R

            B(index)=(unm(i1,k))
           end if
          end if
C
C We define the second equation of the iterative algorithm in terms of the
C nodal values
C
           else
C The row corresponding to the first nodal value of Z^{n,k}
            if (i1.eq.1) then
             v_L=tau/h2
             v_R=tau/h2
             A(index,index)=1.0D+00+tau+(v_L+v_R)
             A(index,index+2)=A(index,index+2)-v_L
             A(index,m+m)=A(index,m+m)-v_R
             v_L=(tau/h2)*lambda
             v_R=(tau/h2)*lambda
             A(index,index+1+fix)=A(index,index+1+fix)
     .                                +(v_L+v_R-tau*mue*theta)
             A(index,index+3+fix)=A(index,index+3+fix)-v_L
             A(index,m+m-1)=A(index,m+m-1)-v_R

             B(index)=(unm(i1,k)+tau*mue*(1.0D+00-theta)*unm(i1,j))

          else
C The row corresponding to the "pre-last" nodal value of Z^{n,k}
           if (i1.eq.nR-nL+1) then
            v_R=(tau/h2)
            v_L=(tau/h2)
            A(index,index)=1.0D+00+tau+(v_L+v_R)
            A(index,index-2)=A(index,index-2)-v_L
            A(index,2)=A(index,2)-v_R
            v_L=(tau/h2)*lambda
            v_R=(tau/h2)*lambda
            A(index,index+1+fix)=A(index,index+1+fix)
     .                                +(v_L+v_R-tau*mue*theta)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L
            A(index,1)=A(index,1)-v_R
```

```
             B(index)=(unm(i1,k)+tau*mue*(1.0D+00-theta)*unm(i1,j))
C The rows corresponding to the rest of the nodal values of Z^{n,k}
          else
            v_L=tau/h2
            v_R=tau/h2
            A(index,index)=1.0D+00+tau+(v_L+v_R)
            A(index,index-2)=A(index,index-2)-v_L
            A(index,index+2)=A(index,index+2)-v_R

            v_L=(tau/h2)*lambda
            v_R=(tau/h2)*lambda
            A(index,index+1+fix)=A(index,index+1+fix)
     .                               +(v_L+v_R-tau*mue*theta)
            A(index,index-1+fix)=A(index,index-1+fix)-v_L
            A(index,index+3+fix)=A(index,index+3+fix)-v_R

            B(index)=(unm(i1,k)+tau*mue*(1.0D+00-theta)*unm(i1,j))
          end if
         end if
        end if

 100        continue
C
C We now call the subroutine DGEFA to factor the matrix A using Gaussian
C elimination. Then we call the subroutine DGESL to solve the system A*X=B
C using the factors computed by DGEFA. The subroutines DGBFA and DGBSL can
C be found in reference [27].
C
        CALL DGEFA(A,lda,lda,ipvt,info)
        CALL DGESL(A,lda,lda,ipvt,b,job)
C
C Check if diff = max {\|W^{n,k}-W^{n,k-1}\|, \|Z^{n,k}-Z^{n,k-1}\|} < tol
C Reset unm(.,1) = W^{n,k} and unm(.,2) = Z^{n,k} for the next time level
C
          diff=0.0D0
          i_max=0
          do 260 i=1,2*(nR-nL+1)
            if (mod(i-1,2).eq.0) then
              i_count=(i-1)/2+1
              if (diff.lt.dabs(un(i_count,1)-B(i))) then
                diff=dabs(un(i_count,1)-B(i))
                i_max=i
              end if
              un(i_count,1)=B(i)
            else
```

```
              i_count=i/2
              if (diff.lt.dabs(un(i_count,2)-B(i))) then
                diff=dabs(un(i_count,2)-B(i))
                i_max=i
              end if
              un(i_count,2)=B(i)
            end if
          B(i)=0.0
 260      continue
C and if diff > tol then repeat iterative loop
          if (mod(nloops,10).eq.0) print *,nloops,diff,i_max
          write(21,*) diff
          if (diff.gt.tol) goto 200
C
C This is the end of the iterative loop
C
          if (mod(loop-1,10).eq.0) then
            print *,p_loop, loop, nloops, diff
          end if
C
C Set W^{n-1}=W^{n,k} and Z^{n-1}=Z^{n,k}
C
          do 270 i=1,nR-nL+1
            unm(i,1)=un(i,1)
            unm(i,2)=un(i,2)
 270      continue
C
C Store total number of iterations in n_tot
C
          n_tot=n_tot+nloops
          nloops=0
          if (mod(loop,1000).eq.0) print *,n_tot
          time=time+tau
          count=count+1
 800    continue
C
C Printing the results:
C We first identify a location for the output according to the size of p_loop
C
        if (p_loop.le.9) then
          write(number1,910) p_loop
          datafile1 =lettert1//number1//'.dat'
          datafile2 =lettert2//number1//'.dat'
        else
          if (p_loop.le.99) then
```

```
            write(number2,920) p_loop
            datafile1 =lettert1//number2//'.dat'
            datafile2 =lettert2//number2//'.dat'
          else
            if (p_loop.le.999) then
              write(number3,930) p_loop
              datafile1 =lettert1//number3//'.dat'
              datafile2 =lettert2//number3//'.dat'
            else
              write(number4,940) p_loop
              datafile1 =lettert1//number4//'.dat'
              datafile2 =lettert2//number4//'.dat'
            end if
          end if
        endif
C
C We store the solutions in the appropriate files (W^{p_loop} and Z^{p_loop}
C will be stored in the data files t1_(p_loop) and t2_(p_loop) respectively)
C
       open(1,status='new',file=datafile1)
       open(2,status='new',file=datafile2)
          do 640 i=1,nR-nL+1
              if (mod(i-1,1).eq.0) then
              write(1,*) un(i,1)
              write(2,*) un(i,2)
              end if
 640      continue
C the following two lines are added to  pint w(L,.):=w(0,1) and z(L,.):=z(0,.)
        write(1,*) un(1,1)
        write(2,*) un(1,2)
      close(1)
      close(2)

 700  continue

 910  format(i1)
 920  format(i2)
 930  format(i3)
 940  format(i4)
      stop
      end
C ---------------------------------------------------
C ---------------------------------------------------
```